2.3: Product and Quotient Rules and Higher Order Derivatives

The Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

"the first times the derivative of the second plus the second times the derivative of the first"

Example 1: Find the derivative of $f(x) = \underbrace{x^7}_{1^{st}}\underbrace{(4x^3)}_{2^{nd}}$.

Product Rule:

$f'(x) = x^7 \frac{d}{dx}(4x^3) + (4x^3)\frac{d}{dx}(x^7)$

$= x^7(12x^2) + 4x^3(7x^6)$

$= 12x^9 + 28x^9 = \boxed{40x^9}$

Using Power Rule:

$f(x) = x^7(4x^3) = 4x^{10}$

$f'(x) = \boxed{40x^9}$

same

Example 2: Find the derivative of $f(x) = (4x^3 + x^2 - 2)(x^4 + 8)$.

$f'(x) = (4x^3 + x^2 - 2)\frac{d}{dx}(x^4 + 8) + (x^4 + 8)\frac{d}{dx}(4x^3 + x^2 - 2)$

$= (4x^3 + x^2 - 2)(4x^3) + (x^4 + 8)(12x^2 + 2x)$

Then clean up

Example 3: Find the derivative of $f(x) = \sqrt{x}(x^5 - 3x^2 + 12x)$.

$f(x) = x^{1/2}(x^5 - 3x^2 + 12x)$

$f'(x) = x^{\frac{1}{2}}\frac{d}{dx}(x^5 - 3x^2 + 12x) + (x^5 - 3x^2 + 12x)\frac{d}{dx}(x^{1/2})$

$= x^{\frac{1}{2}}(5x^4 - 6x + 12) + (x^5 - 3x^2 + 12x)(\frac{1}{2}x^{-1/2})$. Then distribute & clean it up.

(It is more efficient to distribute the $x^{1/2}$ at the beginning, and then you wouldn't need Product Rule)

Example 4: Find the derivative of $y = (4x^3 + 1)(\sqrt{x} + \frac{1}{x} - 2x)$.

Using product rule,

$\frac{dy}{dx} = (4x^3+1)\frac{d}{dx}(x^{1/2} + x^{-1} - 2x) + (x^{\frac{1}{2}} + \frac{1}{x} - 2x)\frac{d}{dx}(4x^3+1)$

$= (4x^3+1)(\frac{1}{2}x^{-1/2} - 1x^{-2} - 2) + (x^{1/2} + \frac{1}{x} - 2x)(12x^2)$

Again, this one is easier to do by distributing, instead of using product rule.

The Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

Example 5: Find the derivative of $f(x)=\dfrac{-x^5}{4x^3}$.

Using Quotient Rule:

$$f'(x)=\frac{4x^3\frac{d}{dx}(-x^5)-(-x^5)\frac{d}{dx}(4x^3)}{(4x^3)^2}$$

$$=\frac{4x^3(-5x^4)+x^5(12x^2)}{16x^6}=\frac{-20x^7+12x^7}{16x^6}=\frac{-8x^7}{16x^6}=\boxed{-\frac{1}{2}x}$$

same!

Using Properties of Exponents and the Power Rule:

$$f(x)=\frac{-x^5}{4x^3}=-\frac{x^2}{4}=-\frac{1}{4}x^2$$

$$f'(x)=-\frac{1}{4}(2x)=\boxed{-\frac{1}{2}x}$$

Example 6: Find the derivative of $g(x)=\cancel{\dfrac{4-2x^3}{x^4+3x^7}}$.

$$g(x)=\frac{4x^2}{x^3+2}$$

$$g'(x)=\frac{(x^3+2)\frac{d}{dx}(4x^2)-4x^2\frac{d}{dx}(x^3+2)}{(x^3+2)^2}$$

$$=\frac{(x^3+2)(8x)-4x^2(3x^2)}{(x^3+2)^2}=\frac{8x^4+16x-12x^4}{(x^3+2)^2}$$

$$=\boxed{\frac{-4x^4+16x}{(x^3+2)^2}}$$

The original Ex 6, $g(x)=\frac{4-2x^3}{x^4+3x^7}$ is in a previous semester's notes. See Archives.

Example 7: Find the derivative of $f(x)=\dfrac{\sqrt{x}}{x^3-x^4}$.

$$f'(x)=\frac{(x^3-x^4)\frac{d}{dx}(x^{1/2})-x^{\frac{1}{2}}\frac{d}{dx}(x^3-x^4)}{(x^3-x^4)^2}$$

$$=\frac{(x^3-x^4)\left(\frac{1}{2}x^{-1/2}\right)-x^{\frac{1}{2}}(3x^2-4x^3)}{(x^3-x^4)^2}=$$

$$=\frac{\frac{1}{2}x^{5/2}-\frac{1}{2}x^{7/2}-3x^{5/2}+4x^{7/2}}{(x^3-x^4)^2}=\frac{-\frac{5}{2}x^{5/2}+\frac{7}{2}x^{7/2}}{(x^3-x^4)^2}\cdot\frac{2}{2}$$

$$\boxed{\frac{-5x^{5/2}+7x^{7/2}}{2(x^3-x^4)^2}}$$

Example 8: Find the derivative of $f(x)=x^2\sin x$.

$$f'(x)=x^2\frac{d}{dx}(\sin x)+(\sin x)\frac{d}{dx}(x^2)$$

$$=x^2\cos x+(\sin x)(2x)$$

$$=\boxed{x^2\cos x+2x\sin x}=\boxed{x(x\cos x+2\sin x)}$$

Example 9: Find the derivative of $f(x)=x+\sin x\cos x$.

$$f'(x)=1+\frac{d}{dx}(\sin x\cos x)$$

$$=1+\sin x\frac{d}{dx}(\cos x)+(\cos x)\frac{d}{dx}(\sin x)$$

$$=1+\sin x(-\sin x)+(\cos x)(\cos x)$$

$$=\underbrace{1-\sin^2 x}_{\cos^2 x}+\cos^2 x=\cos^2 x+\cos^2 x=\boxed{2\cos^2 x}$$

Example 10: Find the derivative of $y=\dfrac{1-\cos x}{\sin x}$.

$$y'=\frac{dy}{dx}=\frac{(\sin x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$=\frac{(\sin x)(0-(-\sin x))-(1-\cos x)(\cos x)}{\sin^2 x}=\frac{(\sin x)(\sin x)-\cos x(1-\cos x)}{\sin^2 x}$$

$\sin^2 x+\cos^2 x=1$

$$=\frac{\sin^2 x-\cos x+\cos^2 x}{\sin^2 x}=\frac{1-\cos x}{\sin^2 x}=\frac{1-\cos x}{1-\cos^2 x}$$

$$=\frac{\cancel{1-\cos x}}{\cancel{(1-\cos x)}(1+\cos x)}=\boxed{\frac{1}{1+\cos x}}$$

Example 11: Prove that $\dfrac{d}{dx}(\csc x)=-\csc x\cot x$.

$$\frac{d}{dx}(\csc x)=\frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$=\frac{\sin x\frac{d}{dx}(1)-1\frac{d}{dx}(\sin x)}{(\sin x)^2}=\frac{(\sin x)(0)-\cos x}{\sin^2 x}$$

$$=\frac{0-\cos x}{\sin^2 x}=-\frac{\cos x}{\sin^2 x}=-\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$=-\csc x\cot x.\ \boxtimes$$

Higher order derivatives:

Once the derivative of $f(x)$ if also a function, it is possible to find the derivative of $f'(x)$ too. This is called the *second derivative* and is denoted $f''(x)$. The second derivative gives the instantaneous rate of change of the derivative. In other words, it tells us how fast the slope is changing.

Similarly, the derivative of $f''(x)$ can be calculated and this is called the *third derivative* $f'''(x)$.

In general, we can keep calculating the derivative of the previous derivative. The *nth derivative* is found by taking the derivative *n* times. The *nth derivative of f* is denoted $f^{(n)}(x)$.

Other notation: $y', y'', y''', \ldots, y^{(n)}$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$$

$$D'y, D^2y, D^3y, \ldots, D^ny$$

Example 12: Suppose that $y = -7x^5 + 6x^4 - \frac{2}{x}$. Find y'''.

$y = -7x^5 + 6x^4 - 2x^{-1}$

The 4th derivative would be denoted $y^{(4)}$

1st derivative: $y' = -35x^4 + 24x^3 + 2x^{-2}$

2nd derivative: $y'' = -140x^3 + 72x^2 - 4x^{-3}$

3rd derivative: $y''' = -420x^2 + 144x + 12x^{-4} = \boxed{-420x^2 + 144x + \frac{12}{x^4}}$

Example 13: Suppose that $f(x) = \sqrt[3]{x}$. Find the second and third derivatives.

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \boxed{\frac{1}{3\sqrt[3]{x^2}}}$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}x^{-\frac{2}{3}-1}\right) = -\frac{2}{9}x^{-5/3} = \boxed{-\frac{2}{9\sqrt[3]{x^5}}}$

$f'''(x) = -\frac{2}{9}\left(-\frac{5}{3}\right)x^{-\frac{5}{3}-1} = \frac{10}{27}x^{-8/3} = \boxed{\frac{10}{27\sqrt[3]{x^8}}}$

Defer until Test 2

Using derivatives to describe the motion of an object:

signed distance (can be negative)

If the dependent variable *t* represents time, and the function $s(t)$ represents the position (distance from a particular point) of an object, then

- the <u>velocity</u> $v(t)$ is the first derivative $s'(t)=\frac{ds}{dt}$.
- the <u>acceleration</u> $a(t)$ is the second derivative $s''(t)=\frac{dv}{dt}$.
- the <u>jerk</u> $j(t)$ is the third derivative $s'''(t)=\frac{da}{dt}$.
- the <u>speed</u> is the absolute value of the velocity $|v(t)|=\left|\frac{ds}{dt}\right|$.

<u>Example 14:</u> The position (in feet) of an object is given by $s(t)=t^4-32t+7$, with t measured in seconds. Find functions representing the (a) velocity, (b) acceleration, and (c) jerk. Find the (d) velocity and acceleration after 1, 2, and 3 seconds.

(e) Find the average velocity over the interval from 1 second to 3 seconds.

(e) <u>Average velocity:</u> $\text{Avg velocity} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$

$$= \frac{s(3)-s(1)}{3-1}$$

$$= \frac{3^4-32(3)+7-(1^4-32(1)+7)}{2}$$

$$= \frac{81-96-1+32}{2} = \frac{-15+31}{2} = \frac{16}{2}$$

$$= \boxed{8 \text{ ft/sec}}$$

<u>For any quantity given by $f(t)$:</u>

Average rate of change in f: $\frac{f(t_2)-f(t_1)}{t_2-t_1}$

See next page

Ex 14 cont'd:

Position: $s(t) = t^4 - 32t + 7$

(a) velocity: $v(t) = s'(t) = 4t^3 - 32$

(b) acceleration: $a(t) = s''(t) = v'(t) = 12t^2$

(c) jerk: $s'''(t) = a'(t) = 24t$

(d) velocity after 1 sec: $v(1) = s'(1) = 4(1)^3 - 32 = -28$ ft/sec

velocity after 2 sec: $v(2) = s'(2) = 4(2)^3 - 32 = 0$ ft/sec

velocity after 3 sec: $v(3) = 4(3)^3 - 32 = 108 - 32 = 76$ ft/sec

acceleration after 1 sec: $a(1) = s''(1) = 12(1)^2 = 12 \frac{\text{ft/sec}}{\text{sec}}$

$= 12$ ft/sec^2

acceleration after 2 sec: $a(2) = s''(2) = 12(2)^2 = 48$ ft/sec^2

acceleration after 3 sec: $a(3) = s''(3) = 12(3)^2 = 108$ ft/sec^2