2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Example 1: Suppose $g(x) = (x^3 + 2)^2$. Find $g'(x)$.

$g(x) = (x^3+2)(x^3+2)$

$= x^6 + 4x^3 + 4$

$\boxed{g'(x) = 6x^5 + 12x^2}$

What if we had $f(x) = (x^3+2)^8$?

or $h(x) = (x^3+2)^{1/2}$?

The chain rule lets us differentiate composite functions.

> The Chain Rule:
>
> If f and g are both differentiable and $F(x) = f(g(x))$, then F is differentiable and
>
> $$F'(x) = f'\big(g(x)\big)g'(x).$$
>
> Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

With the chain rule, we take the derivative of the "outer function" and multiply by the derivative of the "inner function".

Example 2: Find the derivative of $h(x) = (x^3 + 2)^{50}$.

$h'(x) = 50(x^3+2)^{49} \frac{d}{dx}(x^3+2)$

$= 50(x^3+2)^{49}(3x^2)$

$= \boxed{150x^2(x^3+2)^{49}}$

Note:

$y = x^{50}$

$\frac{dy}{dx} = 50x^{49}$

Using Leibniz $\left(\frac{dy}{dx}\right)$ notation:

Let $u = x^3+2$, $y = u^{50}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (50u^{49})(3x^2) = 50(x^3+2)^{49}(3x^2) = \boxed{150x^2(x^3+2)^{49}}$

Example 3: Find the derivative of $f(x) = \sqrt{2x^3 - 5x}$.

$f(x) = (2x^3 - 5x)^{1/2}$

$f'(x) = \frac{1}{2}(2x^3 - 5x)^{-1/2} \frac{d}{dx}(2x^3 - 5x)$

$= \frac{1}{2}(2x^3 - 5x)^{-1/2}(6x^2 - 5) = \boxed{\dfrac{6x^2 - 5}{2\sqrt{2x^3 - 5x}}}$

Example 4: Suppose that $y = \dfrac{2}{(3x^5 - 4)^3}$. Find $\dfrac{dy}{dx}$.

$y = 2(3x^5 - 4)^{-3}$

$\frac{dy}{dx} = -6(3x^5 - 4)^{-4} \frac{d}{dx}(3x^5 - 4)$

$= -6(3x^5 - 4)^{-4}(15x^4)$

$= -90x^4(3x^5 - 4)^{-4} = \boxed{-\dfrac{90x^4}{(3x^5 - 4)^4}}$

Ex: $f(x) = \dfrac{\sqrt{x^2+5}}{7}$

$f(x) = \frac{1}{7}(x^2+5)^{1/2}$

$f'(x) = \frac{1}{7} \cdot \frac{1}{2}(x^2+5)^{-1/2}(2x)$ clean up

Example 5: Find the derivative of $f(x) = \sqrt[3]{\cos x}$.

$f(x) = (\cos x)^{1/3}$

$f'(x) = \frac{1}{3}(\cos x)^{-2/3} \frac{d}{dx}(\cos x)$

$= \frac{1}{3}(\cos x)^{-2/3}(-\sin x) = \boxed{-\dfrac{\sin x}{3\sqrt[3]{\cos^2 x}}}$

Note:
If $y = \cos(x)$
then $\frac{dy}{dx} = -\sin(x)$

Example 6: Find the derivative of $g(x) = \cos \sqrt[3]{x}$.

Note: $y = \frac{1}{\cos x} = \sec x$

$\frac{dy}{dx} = \sec x \tan x$

Example 7: Suppose that $y = \dfrac{1}{\cos x}$. Find $\dfrac{dy}{dx}$.

$y = (\cos x)^{-1}$

$\frac{dy}{dx} = -1(\cos x)^{-2}(-\sin x)$

$= \dfrac{\sin x}{\cos^2 x} = \left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{\cos x}\right)$

$= \boxed{\sec x \tan x}$

Ex $7\frac{1}{2}$: $f(x) = \sin(x^3+7x)$

$f'(x) = \cos(x^3+7x)\frac{d}{dx}(x^3+7x)$

$= \left(\cos(x^3+7x)\right)(3x^2+7)$

$= \boxed{(3x^2+7)\cos(x^3+7x)}$

Ex $7\frac{3}{4}$: $f(x) = \tan\sqrt{x^3-3}$

$f(x) = \tan(x^3-3)^{1/2}$

$f'(x) = \sec^2(x^3-3)^{1/2}\frac{d}{dx}(x^3-3)^{1/2}$

$= \sec^2(x^3-3)^{1/2}\left(\frac{1}{2}\right)(x^3-3)^{-1/2}(3x^2)$

$= \boxed{\dfrac{3x^2\sec^2\sqrt{x^3-3}}{2\sqrt{x^3-3}}}$

Ex $7\frac{7}{8}$ $f(x) = \sec(5x)$

$f'(x) = [\sec(5x)\tan(5x)]\frac{d}{dx}(5x)$

$= [\sec(5x)\tan(5x)](5)$

$= \boxed{5\sec(5x)\tan(5x)}$

Note:
If $y = \sec x$,
$\frac{dy}{dx} = \sec x\tan x$

Example 8: Suppose that $h(x) = (3x^2 - 4)^3 (2x - 9)^2$. Find $h'(x)$.

Product Rule: $h'(x) = (3x^2-4)^3 \frac{d}{dx}(2x-9)^2 + (2x-9)^2 \frac{d}{dx}(3x^2-4)^3$

$$= (3x^2-4)^3 (2)(2x-9)(2) + (2x-9)^2 (3)(3x^2-4)^2 (6x)$$

$$= \underline{4(3x^2-4)^3(2x-9)} + \underline{18x(2x-9)^2(3x^2-4)^2}$$

$$= 2(3x^2-4)^2(2x-9)\left[2(3x^2-4) + 9x(2x-9)\right]$$

$$= 2(3x^2-4)^2(2x-9)\left[6x^2 - 8 + 18x^2 - 81x\right]$$

$$= \boxed{2(3x^2-4)^2(2x-9)(24x^2 - 81x - 8)}$$

Example 9: Suppose that $f(x) = \left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

$$f'(x) = 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{d}{dx}\left(\frac{2x+1}{2x-1}\right)$$

$$= 5\left(\frac{2x+1}{2x-1}\right)^4 \left[\frac{(2x-1)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(2x-1)}{(2x-1)^2}\right]$$

$$= 5\left(\frac{2x+1}{2x-1}\right)^4 \left[\frac{(2x-1)(2) - (2x+1)(2)}{(2x-1)^2}\right] = 5\left(\frac{2x+1}{2x-1}\right)^4 \left[\frac{\cancel{4x} - 2 - \cancel{4x} - 2}{(2x-1)^2}\right]$$

$$= 5\,\frac{(2x+1)^4}{(2x-1)^4}\left(\frac{-4}{(2x-1)^2}\right) = \boxed{\frac{-20(2x+1)^4}{(2x-1)^6}}$$

Example 10: Find the derivative of $y = \cos(\sin(\pi x^2))$.

$$\frac{dy}{dx} = -\sin(\sin(\pi x^2))\frac{d}{dx}(\sin(\pi x^2))$$

$$= -\sin(\sin(\pi x^2))\cos(\pi x^2)\frac{d}{dx}(\pi x^2)$$

$$= -\sin(\sin(\pi x^2))(\cos(\pi x^2))(2\pi x)$$

$$= -2\pi x \sin(\sin(\pi x^2))\cos(\pi x^2)$$

Example 11: Find the derivative of $g(x) = \dfrac{\cos^2 x}{\sqrt{4x+1}}$.

$$g'(x) = \frac{\sqrt{4x+1}\,\frac{d}{dx}(\cos(x))^2 - (\cos(x))^2 \frac{d}{dx}(4x+1)^{1/2}}{(\sqrt{4x+1})^2}$$

$$= \frac{\sqrt{4x+1}\,(2)(\cos x)(-\sin x) - (\cos x)^2\left(\frac{1}{\cancel{2}}\right)(4x+1)^{-1/2}(\cancel{4}^{2})}{4x+1}$$

$$= \frac{-2(\cos x \sin x)\sqrt{4x+1} - 2\cos^2 x \cdot \frac{1}{\sqrt{4x+1}}}{4x+1} \cdot \frac{\sqrt{4x+1}}{\sqrt{4x+1}}$$

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Example 12: Find the first and second derivatives of $f(x) = (x^2+4)^5$.

$$f'(x) = 5(x^2+4)^4 \frac{d}{dx}(x^2+4)$$

$$= 5(x^2+4)^4(2x) = \boxed{10x(x^2+4)^4}$$

$$f''(x) = 10x\frac{d}{dx}(x^2+4)^4 + (x^2+4)^4\frac{d}{dx}(10x)$$

$$= 10x(4)(x^2+4)^3(2x) + (x^2+4)^4(10)$$

$$= 80x^2(x^2+4)^3 + 10(x^2+4)^4$$

(can factor out 10, and also $(x^2+4)^3$)

$$= 10(x^2+4)^3\left[8x^2 + (x^2+4)\right]$$

$$= \boxed{10(x^2+4)^3(9x^2+4)}$$

Example 13: Find the first and second derivatives of $f(x) = \cos(3x^2)$.

$$f'(x) = -\sin(3x^2)\frac{d}{dx}(3x^2)$$

$$= (-\sin(3x^2))(6x) = \boxed{\underbrace{-6x}_{1^{st}}\,\underbrace{\sin(3x^2)}_{2^{nd}}}$$

Need product rule for 2^{nd} derivative:

$$f''(x) = -6x\frac{d}{dx}(\sin(3x^2)) + \sin(3x^2)\frac{d}{dx}(-6x)$$

$$= -6x(\cos(3x^2))\frac{d}{dx}(3x^2) + (\sin(3x^2))(-6)$$

$$= -6x(\cos(3x^2))(6x) - 6\sin(3x^2)$$

$$= \boxed{-36x^2\cos(3x^2) - 6\sin(3x^2)}$$

Ex 11 cont'd

$$g'(x) = \frac{-2(\cos x \sin x)\sqrt{4x+1} - 2\cos^2 x \cancel{\frac{1}{\sqrt{4x+1}}}}{4x+1} \cdot \frac{\sqrt{4x+1}}{\sqrt{4x+1}}$$

$$= \frac{-2(\cos x \sin x)(4x+1) - 2\cos^2 x}{(4x+1)\sqrt{4x+1}}$$

$$= \boxed{\frac{-2(4x+1)\cos x \sin x - 2\cos^2 x}{(4x+1)^{3/2}}}$$

can factor out $-2\cos x$ .... but it doesn't improve much

Example 14: Find the first and second derivatives of $y = \dfrac{2}{(3x+5)^2}$

$$y = 2(3x+5)^{-2}$$

$$y' = -4(3x+5)^{-3}\frac{d}{dx}(3x+5) = 4(3x+5)^{-3}(3)$$

$$= -12(3x+5)^{-3} = \boxed{-\frac{12}{(3x+5)^3}}$$

look at this one for 2nd derivative

$$y'' = 36(3x+5)^{-4}\frac{d}{dx}(3x+5)$$

$$= 36(3x+5)^{-4}(3) = 108(3x+5)^{-4} = \boxed{\frac{108}{(3x+5)^4}}$$

Example 15: Find the equation of the tangent line to $y = \dfrac{x^2}{\sin \pi x + 1}$ at the point where $x = 1$.

Find derivative:

$$y' = \frac{(\sin \pi x + 1)\frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin \pi x + 1)}{(\sin \pi x + 1)^2}$$

$$= \frac{(\sin \pi x + 1)(2x) - x^2\left(\cos(\pi x)\frac{d}{dx}(\pi x) + 0\right)}{(\sin \pi x + 1)^2}$$

$$= \frac{2x(\sin \pi x + 1) - x^2(\cos(\pi x))(\pi)}{(\sin \pi x + 1)^2} = \frac{2x(\sin \pi x + 1) - \pi x^2 \cos(\pi x)}{(\sin \pi x + 1)^2}$$

Slope $= m = y'\Big|_{x=1} = \dfrac{2(1)(\sin(\pi) + 1) - \pi(1)^2\cos(\pi)}{(\sin(\pi) + 1)^2} = \dfrac{2(0+1) - \pi(-1)}{(0+1)^2}$

$$= \frac{2+\pi}{1} = 2 + \pi \leftarrow \text{slope of tangent line!}$$

Find y-value: $y\Big|_{x=1} = \dfrac{1^2}{\sin(\pi(1)) + 1} = \dfrac{1}{0+1} = 1.$

So the point is $(1, 1)$.

Write equation of tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = (2+\pi)(x-1)$$

$$y - 1 = 2x - 2 + \pi x - \pi$$

$$y = 2x + \pi x - 2 - \pi + 1$$

$$\boxed{y = (2+\pi)x - 1 - \pi}$$