2.5: Implicit Differentiation

Example 1: Given the equation $x^3 - 4y - 9x^2 = 5$, find $\frac{dy}{dx}$ by

solving for y:
a) Solving explicitly for y.
b) Implicit differentiation.

(a)
$$x^3 - 9x^2 = 4y + 5$$
$$x^3 - 9x^2 - 5 = 4y$$
$$\frac{x^3 - 9x^2 - 5}{4} = y$$
$$y = \frac{1}{4}(x^3 - 9x^2 - 5)$$
$$\frac{dy}{dx} = \frac{1}{4}(3x^2 - 18x) = \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

(b) Implicit Differentiation
$$x^3 - 4y - 9x^2 = 5$$
$$\frac{d}{dx}(x^3 - 4y - 9x^2) = \frac{d}{dx}(5)$$
$$3x^2 - 4\frac{dy}{dx} - 18x = 0$$
Solve for $\frac{dy}{dx}$: $-4\frac{dy}{dx} = -3x^2 + 18x$
$$\frac{dy}{dx} = \frac{-3x^2}{-4} + \frac{18x}{-4}$$
$$= \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

Example 2: Find $\frac{dy}{dx}$ for $xy = 4$.

(a) solving for y:
$$xy = 4$$
$$y = \frac{4}{x} = 4x^{-1}$$
$$\frac{dy}{dx} = -4x^{-2}$$
$$\boxed{\frac{dy}{dx} = -\frac{4}{x^2}}$$

(b) Implicit diff
$$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$
Need product rule:
$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$
$$x\frac{dy}{dx} + y(1) = 0 \Rightarrow x\frac{dy}{dx} = -y$$
$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Note: If I solve for y & substitute, it matches the $\frac{dy}{dx}$ obtained from solving for y 1st:
$$xy = 4$$
$$y = \frac{4}{x}$$
Put into $\frac{dy}{dx} = -\frac{y}{x}$
$$\frac{dy}{dx} = -\frac{4/x}{x} = -\frac{4}{x}\left(\frac{1}{x}\right)$$
$$= -4/x^2 \checkmark$$

Example 3: Find $\frac{dy}{dx}$ for the equation $x^3y - 2x^2y^3 + x^2 - 3 = 0$.

$$\frac{d}{dx}(\underbrace{x^3y}_{\text{Prod Rule}} - \underbrace{2x^2y^3}_{\text{Prod Rule}} + x^2 - 3) = \frac{d}{dx}(0)$$
$$x^3\frac{d}{dx}(y) + y\frac{d}{dx}(x^3) - 2x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(-2x^2) + 2x - 0 = 0$$
$$x^3\frac{dy}{dx} + y(3x^2) - 2x^2\left(3y^2\frac{dy}{dx}\right) + y^3(-4x) + 2x = 0$$
$$x^3\frac{dy}{dx} + 3x^2y - 6x^2y^2\frac{dy}{dx} - 4xy^3 + 2x = 0$$

See next page

Ex 3 cont'd:

solve for $\frac{dy}{dx}$: $x^3 \frac{dy}{dx} - 6x^2y^2\frac{dy}{dx} = -3x^2y + 4xy^3 - 2x$

$$\frac{dy}{dx}(x^3 - 6x^2y^2) = -3x^2y + 4xy^3 - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2y + 4xy^3 - 2x}{x^3 - 6x^2y^2}}$$

both correct

Note: This is equivalent to $\boxed{\frac{dy}{dx} = \frac{3x^2y - 4xy^3 + 2x}{-x^3 + 6x^2y^2}}$

$$\frac{dy}{dx} = \frac{-3x^2y + 4xy^3 - 2x}{x^3 - 6x^2y^2}\left(\frac{-1}{-1}\right) =$$

could simplify by factoring out x:

$$\frac{dy}{dx} = \frac{x(-3xy + 4y^3 - 2)}{x(x^2 - 6xy^2)} = \frac{-3xy + 4y^3 - 2}{x^2 - 6xy^2}$$

Ex $2\frac{1}{2}$: Find $\frac{dy}{dx}$ for $x^4 - 5y^2 + x^3 - 2y^3 = 7y$

(using implicit differentiation - our only option):

$$\frac{d}{dx}(x^4 - 5y^2 + x^3 - 2y^3) = \frac{d}{dx}(7y)$$

$$4x^3 - 10y\frac{dy}{dx} + 3x^2 - 6y^2\frac{dy}{dx} = 7\frac{dy}{dx}$$

chain Rule

Solve for $\frac{dy}{dx}$: $4x^3 + 3x^2 = 7\frac{dy}{dx} + 10y\frac{dy}{dx} + 6y^2\frac{dy}{dx}$

(Get all terms with $\frac{dy}{dx}$ on one side; all terms without $\frac{dy}{dx}$ on other side)

$$4x^3 + 3x^2 = \frac{dy}{dx}(7 + 10y + 6y^2)$$

Divide both sides by $7 + 10y + 6y^2$:

$$\boxed{\frac{4x^3 + 3x^2}{7 + 10y + 6y^2} = \frac{dy}{dx}}$$

Example 4: Find $\frac{dy}{dx}$ for the equation $\frac{1}{y^4} - \frac{5}{x^4} = 1$.

$$y^{-4} - 5x^{-4} = 1$$

$$\frac{d}{dx}\left(y^{-4} - 5x^{-4}\right) = \frac{d}{dx}(1)$$

$$-4y^{-5}\frac{dy}{dx} + 20x^{-5} = 0$$

$$-4y^{-5}\frac{dy}{dx} = -20x^{-5}$$

$$\frac{dy}{dx} = \frac{-20x^{-5}}{-4y^{-5}} = \frac{20x^{-5}}{4y^{-5}} = \frac{5x^{-5}}{y^{-5}} = \frac{5y^5}{x^5}$$

$$\boxed{\frac{dy}{dx} = \frac{5y^5}{x^5}}$$

Example 5: Find $\frac{dy}{dx}$ for the equation $(x-y)^4 = y^2$.

$$\frac{d}{dx}(x-y)^4 = \frac{d}{dx}(y^2)$$

$$4(x-y)^3\frac{d}{dx}(x-y) = 2y\frac{dy}{dx}$$

$$4(x-y)^3\left(1 - \frac{dy}{dx}\right) = 2y\frac{dy}{dx}$$

$$4(x-y)^3 - 4(x-y)^3\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$4(x-y)^3 = 4(x-y)^3\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$4(x-y)^3 = \frac{dy}{dx}\left[4(x-y)^3 + 2y\right]$$

$$\frac{4(x-y)^3}{4(x-y)^3 + 2y} = \frac{dy}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{2(x-y)^3}{2(x-y)^3 + y}}$$

(dividing out a 2)

Example 6: Find $\frac{dy}{dx}$ for the equation $x + \cos(x^2 y) = y$

$$\frac{d}{dx}\left(x + \cos(x^2y)\right) = \frac{d}{dx}(y)$$

$$1 - \sin(x^2y)\frac{d}{dx}(x^2y) = \frac{dy}{dx}$$

need product rule!

$$1 - \sin(x^2y)\left[x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2)\right] = \frac{dy}{dx}$$

$$1 - \sin(x^2y)\left[x^2\frac{dy}{dx} + y(2x)\right] = \frac{dy}{dx}$$

$$1 - \left(\sin(x^2y)\right)x^2\frac{dy}{dx} - \left(\sin(x^2y)\right)(2xy) = \frac{dy}{dx}$$

$$1 - x^2\frac{dy}{dx}\sin(x^2y) - 2xy\sin(x^2y) = \frac{dy}{dx}$$

$$1 - 2xy\sin(x^2y) = x^2\frac{dy}{dx}\sin(x^2y) + \frac{dy}{dx}$$

$$1 - 2xy\sin(x^2y) = \frac{dy}{dx}\left(x^2\sin(x^2y) + 1\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 2xy\sin(x^2y)}{1 + x^2\sin(x^2y)}}$$

or (multiply by $\frac{-1}{-1}$)

$$\boxed{\frac{dy}{dx} = \frac{-1 + 2xy\sin(x^2y)}{-1 - x^2\sin(x^2y)}}$$

<u>Finding higher-order derivatives using implicit differentiation</u>:

To find the second derivative, denoted $\frac{d^2y}{dx^2}$, differentiate the first derivative $\frac{dy}{dx}$ with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 7: Find $\frac{d^2y}{dx^2}$ for the equation $x^3 - 2x^2 = y$.

Find 1st derivative:

$$\frac{d}{dx}\left(x^3 - 2x^2\right) = \frac{d}{dx}(y)$$

$$3x^2 - 4x = \frac{dy}{dx}$$

2nd derivative:

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{d^2y}{dx^2} = \boxed{6x - 4}$$

Example 8: Find $\frac{d^2y}{dx^2}$ for the equation $xy^2 - y = 3$.

1st derivative:

$$\frac{d}{dx}\left(xy^2 - y\right) = \frac{d}{dx}(3)$$

$$\underbrace{x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)}_{\text{Product Rule}} - \frac{dy}{dx} = 0$$

$$x \cdot 2y\frac{dy}{dx} + y^2(1) - \frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} - \frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx}(2xy - 1) = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{2xy-1}$$

multiply by $\frac{-1}{-1}$

$$\frac{dy}{dx} = \frac{y^2}{-2xy+1}$$

$$\frac{d^2y}{dx^2} = \frac{(-2xy+1)\frac{d}{dx}(y^2) - y^2\frac{d}{dx}(-2xy+1)}{(-2xy+1)^2}$$

$$= \frac{(-2xy+1)2y\frac{dy}{dx} - y^2\left(-2x\frac{d}{dx}(y) + y\frac{d}{dx}(-2x) + 0\right)}{(-2xy+1)^2}$$

$$= \frac{(-2xy+1)2y\frac{dy}{dx} - y^2\left(-2x\frac{dy}{dx} + y(-2)\right)}{(-2xy+1)^2}$$

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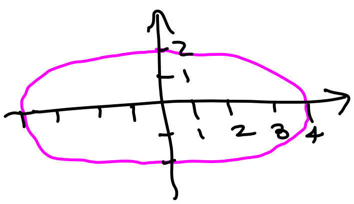
Std form for ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Example 9: Find the equation of the tangent line to the ellipse $4x^2 + 16y^2 = 64$ at the point $(2, \sqrt{3})$.

$4x^2 + 16y^2 = 64$

$\frac{4x^2}{64} + \frac{16y^2}{64} = \frac{64}{64}$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$



x-direction: $\sqrt{16} = 4$

y-direction: $\sqrt{4} = 2$

$\frac{d}{dx}(4x^2 + 16y^2) = \frac{d}{dx}(64)$

$8x + 32y\frac{dy}{dx} = 0$

$32y\frac{dy}{dx} = -8x$

$\frac{dy}{dx} = \frac{-8x}{32y} = -\frac{x}{4y}$

Slope: $m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{3})} = -\frac{2}{4\sqrt{3}} = -\frac{1}{2\sqrt{3}}$

$= -\frac{\sqrt{3}}{6}$

Find eqn:

$y - y_1 = m(x - x_1)$

$y - \sqrt{3} = -\frac{\sqrt{3}}{6}(x - 2)$

$y = -\frac{\sqrt{3}}{6}x + \frac{2\sqrt{3}}{6} + \sqrt{3}$

$y = -\frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{3} + \frac{3\sqrt{3}}{3}$

$\boxed{y = -\frac{\sqrt{3}}{6}x + \frac{4\sqrt{3}}{3}}$

<u>Definition</u>: Two curves are said to be *orthogonal* if at each point of intersection, their tangent lines are perpendicular.

Example 10: Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $4x^2 + 9y^2 = 72$ are orthogonal.

Find slopes of tangent lines:

$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(5)$

$2x - 2y\frac{dy}{dx} = 0$

$-2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{-2y}$

$\frac{dy}{dx} = \frac{x}{y}$

$\frac{d}{dx}(4x^2 + 9y^2) = \frac{d}{dx}(72)$

$8x + 18y\frac{dy}{dx} = 0$

$18y\frac{dy}{dx} = -8x$

$\frac{dy}{dx} = \frac{-8x}{18y}$

$\frac{dy}{dx} = -\frac{4x}{9y}$

Find the intersection points:

$x^2 - y^2 = 5$

$x^2 = y^2 + 5$

substitute $x^2 = y^2 + 5$ into other eqn:

$4x^2 + 9y^2 = 72$

$4(y^2 + 5) + 9y^2 = 72$

$4y^2 + 20 + 9y^2 = 72$

$13y^2 = 52$

$y^2 = 4$

$y = \pm 2$

$y = 2 \Rightarrow x^2 = y^2 + 5$

$x^2 = 2^2 + 5$

$= 9 \Rightarrow x = \pm 3$

$y = -2 \Rightarrow x^2 = (-2)^2 + 5$

$= 9 \Rightarrow x = \pm 3$

intersection pts:

$(3, 2)$, $(3, -2)$

$(-3, 2)$, $(-3, -2)$

See next Page

Ex 8 continued

$$\frac{d^2y}{dx^2} = \frac{(-2xy+1)2y\frac{dy}{dx} - y^2\left(-2x\frac{dy}{dx} + y(-2)\right)}{(-2xy+1)^2}$$

$$= \frac{(-2xy+1)2y\frac{dy}{dx} + 2xy^2\frac{dy}{dx} + 2y^3}{(-2xy+1)^2}$$

Substitute $\frac{dy}{dx} = \frac{y^2}{-2xy+1}$:

$$\frac{d^2y}{dx^2} = \frac{(-2xy+1)2y\left(\frac{y^2}{-2xy+1}\right) + 2xy^2\left(\frac{y^2}{-2xy+1}\right) + 2y^3}{(-2xy+1)^2} \cdot \frac{-2xy+1}{-2xy+1}$$

$$= \frac{(-2xy+1)(2y^3) + 2xy^4 + 2y^3(-2xy+1)}{(-2xy+1)^3}$$

$$= \frac{-4xy^4 + 2y^3 + 2xy^4 - 4xy^4 + 2y^3}{(-2xy+1)^3} = \boxed{\frac{-6xy^4 + 4y^3}{(-2xy+1)^3}}$$

Ex 10 cont'd: Find slopes at each intersection pt:

At (3,2): hyperbola:

$$\left.\frac{dy}{dx}\right|_{(3,2)} = \left.\frac{x}{y}\right|_{(3,2)} = \frac{3}{2}$$

ellipse:

$$\left.\frac{dy}{dx}\right|_{(3,2)} = \left.\frac{-4x}{9y}\right|_{(3,2)} = -\frac{4(3)}{9(2)} = -\frac{12}{18} = -\frac{2}{3}$$

opposite reciprocals, so curves are orthogonal.

At (−3,2): hyp $\left.\frac{dy}{dx}\right|_{(-3,2)} = \left.\frac{x}{y}\right|_{(-3,2)} = \frac{-3}{2}$

ellipse $\left.\frac{dy}{dx}\right|_{(-3,2)} = -\left.\frac{4x}{9y}\right|_{(-3,2)} = \frac{-4(-3)}{9(2)} = \frac{12}{18} = \frac{2}{3}$

opp reciprocals.

At (−3, 2):

$\frac{x}{y} = \frac{-3}{2}$

$-\frac{4x}{9y} = \frac{-4(-3)}{9(2)} = \frac{2}{3}$

similar for (−3, −2).