2.6: Related rates

General idea for solving rate problems:

1. Draw a sketch if applicable. The only dimensions you put on your sketch should be those that do not change.
2. Write down, in calculus notation, the rates you know and want. (also when)
3. Write an equation relating the quantities that are changing. ← the variables from Know & Want
4. Differentiate it implicitly, with respect to time.
5. Substitute known quantities.
6. Solve for the required rate.

when $r = 8''$, $\boxed{\frac{dV}{dt} = 512\pi \text{ in}^3/\text{min}}$

Example 1: The radius of a sphere is increasing at the rate of 2 inches per minute. Find the rate of change in volume when the radius is 6 inches. What about when radius is 8''?

Know: $\frac{dr}{dt} = 2\text{in/min}$

Volume of Sphere: $V = \frac{4}{3}\pi r^3$ (memorize!)

Want: $\frac{dV}{dt}$

When: $r = 6''$, $r = 8''$

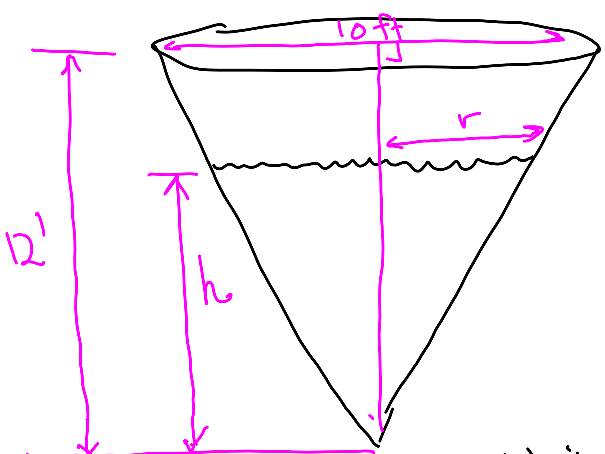
$$V = \frac{4}{3}\pi r^3$$
$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$
$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

Substitute $r = 6''$, $\frac{dr}{dt} = 2\text{in/min}$:

$$\frac{dV}{dt} = 4\pi(6\text{in})^2\left(\frac{2\text{in}}{\text{min}}\right) = \boxed{288\pi \text{ in}^3/\text{min}}$$

when $r = 6''$

Example 2: A conical tank (with vertex down) has a diameter of 10 feet at the top and is 12 feet deep. If water is draining out at 10 cubic feet per minute, what is the rate of change in depth when the water is 8 feet deep?



Know: $\frac{dV}{dt} = -\frac{10\text{ft}^3}{\text{min}}$ V = volume of water

Want: $\frac{dh}{dt}$

When: $h = 8$ ft

Writing eqn that relates V to h:

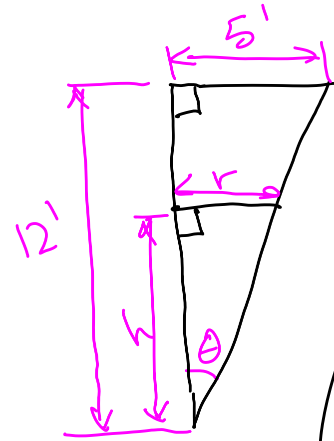
$$V = \frac{1}{3}\pi r^2 h$$ (memorize this!)

Need to get rid of r (because r does not appear in my Know or Want)

next page

Relating r to h:

Similar Triangles:



$$\frac{12'}{h} = \frac{5'}{r}$$

OR

$$\tan\theta = \frac{r}{h} = \frac{5'}{12'}$$

Solve for r:

$$12r = 5h$$

$$r = \frac{5}{12}h$$

Substitute $r = \frac{5}{12}h$ into $V = \frac{1}{3}\pi r^2 h$:

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$$

$$V = \frac{1}{3}\pi \frac{25}{144} h^3$$

$$V = \frac{25\pi}{432} h^3$$

Differentiate w/respect to t:

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{25\pi}{432} h^3\right)$$

$$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{75\pi}{432} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{432}{75\pi h^2} \cdot \frac{dV}{dt}$$

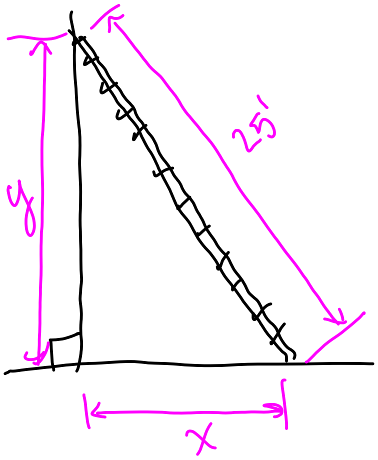
Substitute: $\frac{dV}{dt} = -10\frac{ft^3}{min}$, $h = 8ft$

$$\frac{dh}{dt} = \frac{432}{75\pi (8ft)^2} \cdot \left(\frac{-10 ft^3}{min}\right)$$

$$\frac{dh}{dt} = \frac{-4320 \quad ft^3}{4800\pi \ ft^2 \cdot min}$$

$$= \boxed{-\frac{9}{10\pi} \ ft/min}$$

Example 3: A ladder 25 feet long is leaning against a wall. The base of the ladder is pulled away at 2 feet per second. How fast is the top of the ladder moving when the base is 9 feet away? What about when it is 24 feet away?



Know: $\frac{dx}{dt} = +2$ ft/sec

Want: $\frac{dy}{dt}$

when: $x = 9$ ft , $x = 24$ ft

Write an eqn that relates x and y:

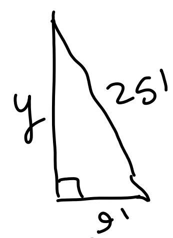
Pythagorean Thm:

$x^2 + y^2 = 25^2$

$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(625)$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

We know x and $\frac{dx}{dt}$, need to find y when $x = 9'$



$y^2 + 9^2 = 25^2$

$y^2 + 81 = 625$

$y^2 = 544$

$y = \sqrt{544}$

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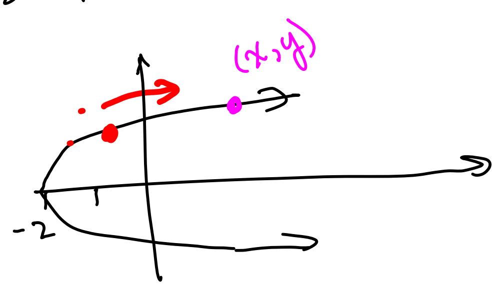
Example 4: A particle is moving along the parabola $y^2 = 4x + 8$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. How fast is the x-coordinate changing at this instant?

$y^2 = 4x + 8$

Variation of: $y^2 = x$

$y^2 = 4(x+2)$

shift it left 2



Need eqn relating x to y:

$y^2 = 4x + 8$

$\frac{d}{dt}(y^2) = \frac{d}{dt}(4x+8)$

$2y\frac{dy}{dt} = \frac{4\,dx}{dt} + 0$

Know: $\frac{dy}{dt} = +3$ units/sec

Want: $\frac{dx}{dt}$

when: $x = 7$, $y = 6$

Solve for $\frac{dx}{dt}$:

$\frac{2y}{4} \cdot \frac{dy}{dt} = \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{y}{2} \cdot \frac{dy}{dt}$

Substitute $x = 7$, $y = 6$, $\frac{dy}{dt} = 3 \frac{\text{units}}{\text{sec}}$

$\frac{dx}{dt} = \frac{6}{2} \cdot 3$ units/sec

$= \boxed{9 \text{ units/sec}}$

Ex 3 cont'd

when $x = 9'$, $y = \sqrt{544}$ ft

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Substitute: $x = 9'$; $y = \sqrt{544}'$, $\frac{dx}{dt} = 2$ ft/sec

$$2(9\,ft)(2\,ft/sec) + 2(\sqrt{544})ft\,\frac{dy}{dt} = 0$$

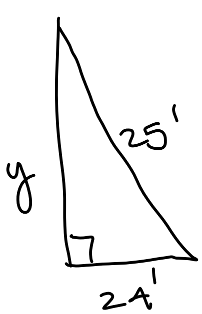
$$36\,ft^2/sec + 2\sqrt{544}\,ft\,\frac{dy}{dt}$$

$$2\sqrt{544}\,ft\,\frac{dy}{dt} = -36\,ft^2/sec$$

$$\frac{dy}{dt} = \frac{-36\,ft^2/sec}{2\sqrt{544}\,ft}$$

$$\approx \boxed{-0.7717\ ft/sec}$$

when $x = 24'$:



$$24^2 + y^2 = 25^2$$

$$576 + y^2 = 625$$

$$y^2 = 49$$

$$y = 7$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$x = 24'$; $y = 7'$, $\frac{dx}{dt} = 2$ ft/sec

$\Downarrow$

$$2(24'ft)(2\,ft/sec) + 2(7')\frac{dy}{dt} = 0$$

$$96\,ft^2/sec = -14\,ft\,\frac{dy}{dt}$$

$$\frac{96\,ft^2/sec}{-14\,ft} = \frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{48}{7}\,ft/sec$$

$$= \boxed{-6\tfrac{6}{7}\ ft/sec}$$

This problem would be easier if I solved for $\frac{dy}{dt}$ before substituting:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

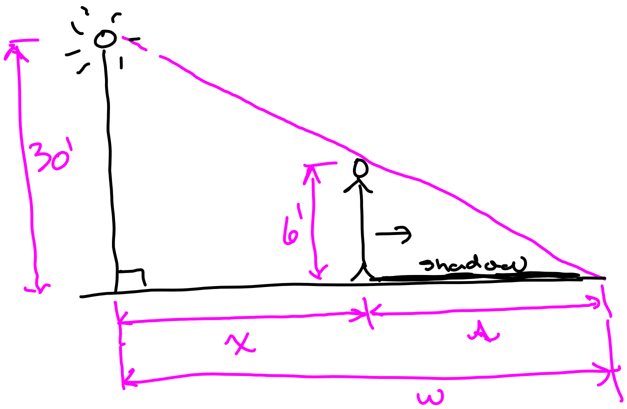
$$\frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Then,

$$\left.\frac{dy}{dt}\right|_{x=9,\,y=\sqrt{544}} = -\frac{9'}{\sqrt{544}'}(2\,ft/sec) = \boxed{-\frac{18}{\sqrt{544}}\ ft/sec}$$

$$\left.\frac{dy}{dt}\right|_{x=24,\,y=7} = -\frac{24}{7}(2\,ft/sec) = \boxed{-\frac{48}{7}\,ft/sec}$$

Example 5: A 6-foot tall man walks away from a 30-foot tall lamppost at a speed of 400 feet per minute. When he is 50 feet away from the lamppost, at what rate is his shadow lengthening? How fast is the tip of his shadow moving?



Know: $\frac{dx}{dt} = +400$ ft/min

want: $\frac{ds}{dt}$, $\frac{dw}{dt}$

when: $50\text{ft} = x$

Need a relationship (equation) between x and either w or s

Could write $w = x + s$, but this gives 2 unknown rates.

similar triangles:

$\frac{30}{6} = \frac{w}{s}$ or $\frac{6}{s} = \frac{30}{w}$

Must incorporate the x; replace w by $x+s$:

$\frac{30}{6} = \frac{x+s}{s}$

$5 = \frac{x}{s} + 1$

$5s = x + s$

$4s = x$

$\frac{d}{dt}(4s) = \frac{d}{dt}(x)$

$4\frac{ds}{dt} = \frac{dx}{dt}$

Substitute: $\frac{dx}{dt} = 400$ ft/min

$4\frac{ds}{dt} = 400$ ft/min

$\frac{ds}{dt} = 100$ ft/min

The shadow is lengthening at 100 ft/min.

Back to

$w = x + s$

$\frac{d}{dt}(w) = \frac{d}{dt}(x+s)$

$\frac{dw}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$

Substitute

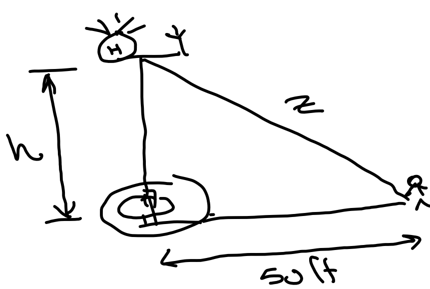
$\frac{dx}{dt} = 400$ ft/min, $\frac{ds}{dt} = 100$ ft/min

$\Rightarrow \frac{dw}{dt} = 400$ ft/min $+ 100$ ft/min

$= 500$ ft/min

Tip of shadow moving at 500 ft/min

Example 6: At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport. The helicopter is rising vertically at a speed of 44 ft/second. How fast is the distance between the helicopter and the man changing when it is at an altitude of 120 ft?



Know: $\frac{dh}{dt} = +44$ ft/sec

want: $\frac{dz}{dt}$

when: $h = 120$ ft

Need an equation relating h and z:

$h^2 + (50)^2 = z^2$

$\frac{d}{dt}(h^2 + 2500) = \frac{d}{dt}(z^2)$

$2h\frac{dh}{dt} + 0 = 2z\frac{dz}{dt}$

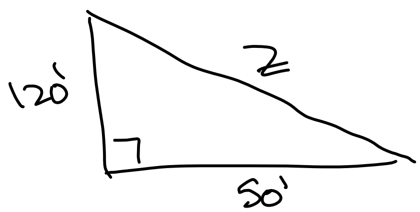
$2h\frac{dh}{dt} = 2z\frac{dz}{dt}$

See next page

Ex 6 cont'd:

Substitute $h = 120'$, $\frac{dh}{dt} = 44\ ft/sec$ ... what about z?

Draw a new picture, that applies only at the instant $h = 120\,ft$:



$$50^2 + 120^2 = z^2$$

$$z^2 = 16900$$

$$z = 130'$$

$$2h\frac{dh}{dt} = 2z\frac{dz}{dt}$$

$$2(120')(44\ ft/sec) = 2(130')\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{2(120\ ft)(44\ ft/sec)}{2(130\ ft)} \approx \boxed{40.615\ ft/sec}$$