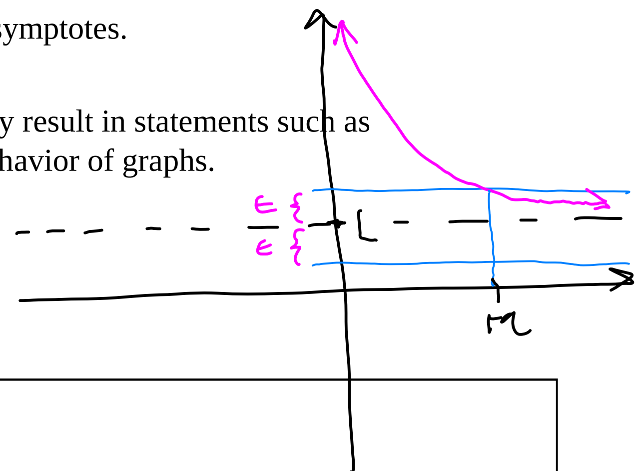
3.5: Limits at Infinity

There are two types of limits involving infinity.

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes.

Infinite limits (covered in Section 1.5) take the form of statements like $\lim_{x\to a} f(x)=\infty$ or $\lim_{x\to a} f(x)=-\infty$. Infinite limits generally result in vertical asymptotes.

When combined, these two types of limits involving infinity result in statements such as $\lim_{x\to\infty} f(x)=\infty$ or $\lim_{x\to\infty} f(x)=-\infty$, which describe the end behavior of graphs.



Limits at infinity:

Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x)=L \qquad \lim_{x\to\infty} f(x)=L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

More precisely, $\lim_{x\to\infty} f(x)=L$ if, for every number $\varepsilon>0$, there exists a corresponding number $M>0$ such that for all x, $|f(x)-L|<\varepsilon$ whenever $x>M$.

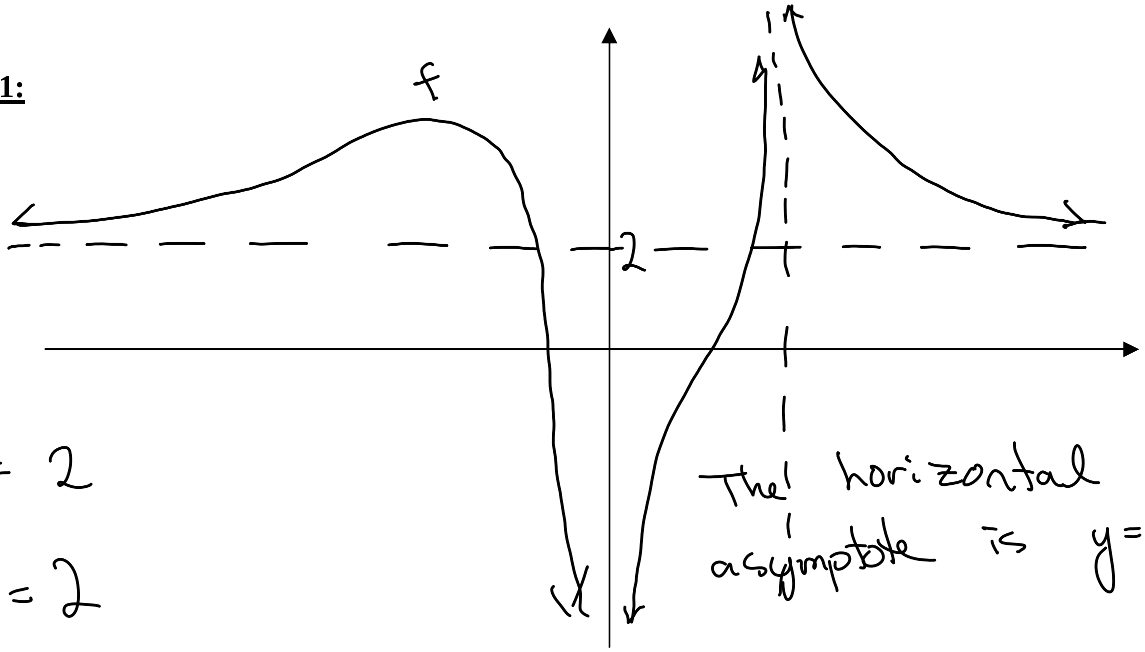
Let f be a function defined on some interval $(-\infty,a)$. Then

$$\lim_{x\to-\infty} f(x)=L \qquad \lim_{x\to-\infty} f(x)=L$$

means that the values of $f(x)$ can be made arbitrarily close to L by making x a sufficiently large negative number.

More precisely, $\lim_{x\to\infty} f(x)=L$ if, for every number $\varepsilon>0$, there exists a corresponding number $N<0$ such that for all x, $|f(x)-L|<\varepsilon$ whenever $x<N$.

Example 1:



Here,

$\lim_{x \to \infty} f(x) = 2$

$\lim_{x \to -\infty} f(x) = 2$

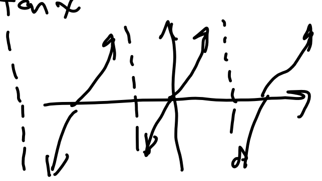
The horizontal asymptote is $y = 2$.

Horizontal asymptotes:

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L .$$

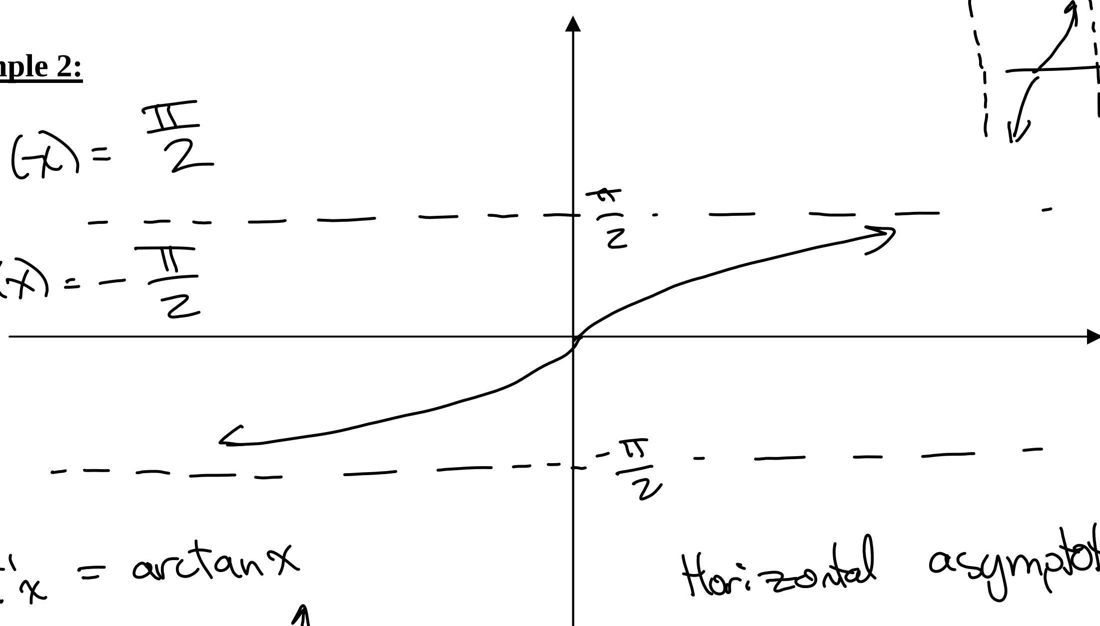
$y = \tan x$



Example 2:

$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

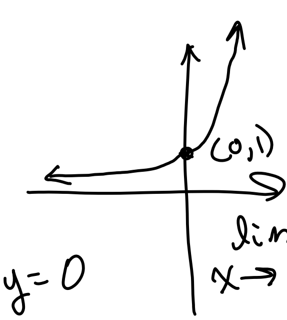


$y = \tan^{-1} x = \arctan x$

Horizontal asymptotes are

$y = \pm \frac{\pi}{2}$

Ex: $y = e^x$



horizontal asymptote $y = 0$

$\lim_{x \to -\infty} e^x = 0$

Example 3: Determine $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.

$\boxed{\lim_{x\to\infty}\frac{1}{x}=0}$ One could also say this: $\boxed{\text{As } x\to\infty,\ \frac{1}{x}\to 0}$

As $x\to+\infty$, $\frac{1}{x}\to\frac{1}{+\text{huge}}\to+\text{tiny}\to 0$

$\boxed{\lim_{x\to-\infty}\frac{1}{x}=0}$

As $x\to-\infty$, $\frac{1}{x}\to\frac{1}{-\text{huge}}\to-\text{tiny}\to$

Theorem: If $r>0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r}=0$. Also $\lim_{x\to-\infty}\frac{1}{x^r}=0$ if x^r is defined for all *x*.

Note: $\lim_{x\to\infty}\frac{a}{x^r}=a\lim_{x\to\infty}\left(\frac{1}{x^r}\right)=a(0)=0$

Example 4: Determine $\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right)$

$\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right)=9+0=\boxed{9}$

As $x\to\infty$,

$9+\frac{3}{x^4}\to 9+\frac{3}{(+\text{huge})^4}$

$\to 9+\text{tiny}$

$\to 9+0=9$

$\lim_{x\to\infty}\left(\frac{\frac{3}{x}-\frac{2x}{x}}{\frac{4x}{x}+\frac{6}{x}}\right)$

To take advantage of this theorem when determining the limit at infinity of a quotient, divide numerator and denominator by the largest power of the variable in the denominator.

Example 5: Find $\lim_{x\to\infty}\frac{3-2x}{4x+6}$.

$\lim_{x\to\infty}\left(\frac{3-2x}{4x+6}\right)=\lim_{x\to\infty}\left(\frac{3-2x}{4x+6}\right)\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)=\lim_{x\to\infty}\left(\frac{\frac{3}{x}-2}{4+\frac{6}{x}}\right)$

$=\frac{0-2}{4+0}=\frac{-2}{4}=\boxed{-\frac{1}{2}}$

Example 6: Find $\lim_{x\to\infty}\frac{2x^2+15x+9}{5x^3-14}$ and $\lim_{x\to-\infty}\frac{2x^2+15x+9}{5x^3-14}$.

$$\lim_{x\to\infty}\frac{2x^2+15x+9}{5x^3-14} = \lim_{x\to\infty}\frac{\frac{2x^2}{x^3}+\frac{15x}{x^3}+\frac{9}{x^3}}{\frac{5x^3}{x^3}-\frac{14}{x^3}}$$

$$= \lim_{x\to\infty}\frac{\frac{2}{x}+\frac{15}{x^2}+\frac{9}{x^3}}{5-\frac{14}{x^3}} = \frac{0+0+0}{5-0} = \frac{0}{5} = \boxed{0}$$

Would anything change for $x\to-\infty$? No. Theorem says all these limits are still 0.

So, $\lim_{x\to-\infty}\frac{2x^2+15x+9}{5x^3-14} = \boxed{0}$ also.

Example 7: Find the horizontal asymptote (if any) of $h(x)=\frac{8x^2-6x+1}{3x^2+4x-5}$.

$$\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty}\frac{\frac{8x^2}{x^2}-\frac{6x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{4x}{x^2}-\frac{5}{x^2}}$$

$$= \lim_{x\to\pm\infty}\frac{8-\frac{6}{x}+\frac{1}{x^2}}{3+\frac{4}{x}-\frac{5}{x^2}} = \frac{8-0+0}{3+0-0}$$

$$= \frac{8}{3}$$

$$\boxed{\text{The horizontal asymptote is } y=\frac{8}{3}.}$$

Example 8: Determine $\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2}$ and $\lim_{x\to-\infty}\frac{7x^5-5x+1}{8-x^2}$.

$$\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2} = \lim_{x\to\infty}\frac{\frac{7x^5}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}{\frac{8}{x^2}-\frac{x^2}{x^2}}$$

$$= \lim_{x\to\infty}\frac{7x^3-\frac{5}{x}+\frac{1}{x^2}}{\frac{8}{x^2}-1} = \frac{\lim_{x\to\infty}(7x^3)-\lim_{x\to\infty}\left(\frac{5}{x}\right)+\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}\left(\frac{8}{x^2}\right)-\lim_{x\to\infty}(1)}$$

$$= \frac{\lim_{x\to\infty}(7x^3)-0+0}{0-1} = -\lim_{x\to\infty}(7x^3) = \boxed{-\infty}$$ (this limit does not exist)

As $x\to+\infty$, $-7x^3\to -7(+\text{huge})^3 \to -7(+\text{huge}) \to -\text{huge}$

See next page

Now consider $x \to -\infty$:

$$\lim_{x\to-\infty} \frac{7x^5-5x+1}{8-x^2} = \lim_{x\to-\infty}(-7x^3) = \boxed{+\infty}$$ (limit does not exist)

Scratch

As $x \to -\infty$, $-7x^3 \to -7(-\text{huge})^3$

$\to -7(-\text{huge})$

$\to +\text{huge} \to +\infty$

Example 9: Determine $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x}-\sqrt{x}}{8-x^{-2}}$

$$\lim_{x\to\infty}\frac{\sqrt[3]{x}-\sqrt{x}}{8-x^{-2}} = \lim_{x\to\infty}\frac{\sqrt[3]{x}-\sqrt{x}}{8-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{x^{\frac13}-x^{\frac12}}{8-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^{\frac13}\left(1-x^{\frac12-\frac13}\right)}{8-\frac{1}{x^2}}$$

Note: $\infty - \infty$ is an indeterminate form

$$= \lim_{x\to\infty}\frac{x^{\frac13}\left(1-x^{1/6}\right)}{8-\frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left[x^{1/3}\left(1-x^{1/6}\right)\right]}{8-0} = \boxed{-\infty}$$

As $x \to +\infty$

$\sqrt[3]{x}\left(1-\sqrt[6]{x}\right)$

$\sqrt[3]{+\text{huge}}\left(1-\sqrt[6]{+\text{huge}}\right)$

$+\text{huge}\,(1-\text{huge})$

$+\text{huge}\,(-\text{huge})$

$-\text{huge}$

Divide by 8 still

$-\text{huge}$

Example 10: Evaluate the limit of $f(x)=\dfrac{x^3+4x^2-11x-7}{x^2+6x+1}$ as x approaches $\pm\infty$.

$$\lim_{x\to\infty}\frac{x^3+4x^2-11x-7}{x^2+6x+1} = \lim_{x\to\infty}\left[\frac{\frac{x^3}{x^2}+\frac{4x^2}{x^2}-\frac{11x}{x^2}-\frac{7}{x^2}}{\frac{x^2}{x^2}+\frac{6x}{x^2}+\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x+4-\frac{11}{x}-\frac{7}{x^2}}{1+\frac{6}{x}+\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x+4-0-0}{1+0+0}\right] = \lim_{x\to\infty}(x+4) = \boxed{\infty}$$ (doesn't exist)

$$\lim_{x\to-\infty}\frac{x^3+4x^2-11x-7}{x^2+6x+1} = \lim_{x\to-\infty}(x+4) = \boxed{-\infty}$$

has slant asymptote -- see p. 8

Example 11: Evaluate the limit of $f(x)=\dfrac{8x^2-7x+1}{2x-3}$ as x approaches $\pm\infty$.

$$\lim_{x\to\infty}f(x) = \lim_{x\to\infty}\frac{8x-7+\frac{1}{x}}{2-\frac{3}{x}} = \lim_{x\to\infty}\left(\frac{8x-7}{2}\right) = +\infty$$

$$\lim_{x\to-\infty}f(x) = \lim_{x\to-\infty}\left(\frac{8x-7}{2}\right) = -\infty$$

$y = 4x - \frac{7}{2}$ (linear) ... so it has a slant asymptote

See page 8 for how to find the slant asymptote

Slant asymptotes:

> Note:
> The graphs of the functions in the previous two examples have *oblique (slant) asymptotes*. This is because the function values (y-values) approached those of a linear function $y = mx + b$ as x approached $\pm\infty$.

For rational functions, the function has a slant asymptote if the degree of the numerator is 1 more than the degree of the denominator.

Example 12: Evaluate the limit of $f(x) = \dfrac{2x^3 - x^2 + x}{x-3}$ as x approaches $\pm\infty$. Does the graph of this function have a slant asymptote?

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{2x^3}{x} - \frac{x^2}{x} + \frac{x}{x}}{\frac{x}{x} - \frac{3}{x}} = \lim_{x\to\infty} \frac{2x^2 - x + 1}{1 - \frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{2x^2 - x + 1}{1 - 0} = \lim_{x\to\infty} (2x^2 - x + 1)$$

$$= \lim_{x\to\infty} (2x^2 - x) + \lim_{x\to\infty} (1)$$

$$= \lim_{x\to\infty} (x(2x-1)) + 1$$

$$= \boxed{+\infty}$$

Note: As $x \to \infty$,
$x(2x-1)+1 \to +\text{huge}(+\text{huge} - 1) + 1$
$\to +\text{huge}(+\text{huge}) \to +\text{huge}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x(2x-1)) + 1$$

$$= \boxed{+\infty}$$

(those limits do not exist.)

As $x \to -\infty$,
$x(2x-1)+1 \to -\text{huge}(-\text{huge} - 1) + 1$
$\to -\text{huge}(-\text{huge}) \to +\text{huge}$

see next page

Ex 12 cont'd: This function does not have a slant asymptote. Because deg(numerator) = deg(denom) + 2, I would expect to get a quadratic (degree 2) for the polynomial long division. The graph will approach the graph of this quadratic as $x \to \pm\infty$.

$$
\begin{array}{r}
2x^2 + 5x + 16 \\
x-3 \overline{\big)\, 2x^3 - x^2 + x + 0} \\
-(2x^3 - 6x^2) \\
\hline
5x^2 + x \\
-(5x^2 - 15x) \\
\hline
16x + 0 \\
-(16x - 48) \\
\hline
48
\end{array}
$$

Check by multiplying:

$(x-3)(2x^2+5x+16) + 48$

$= 2x^3 + 5x^2 + 16x - 6x^2 - 15x - 48 + 48$

$= 2x^3 - x^2 + x$ ✓

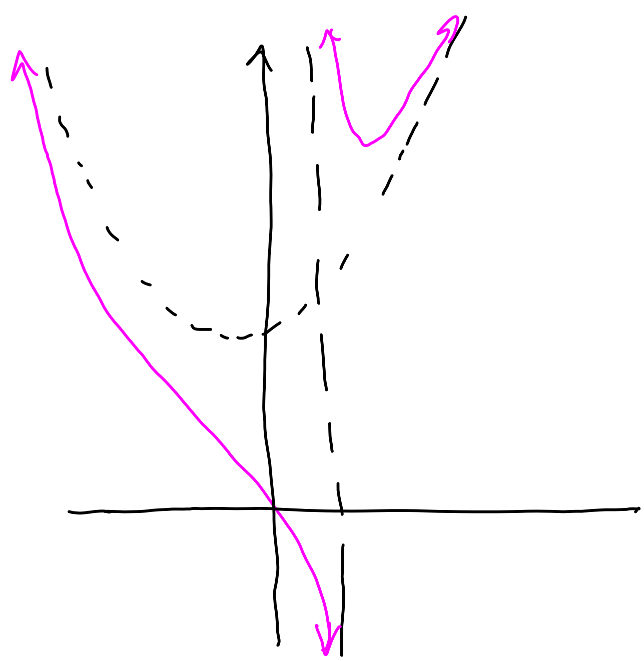
Thus

$f(x) = 2x^2 + 5x + 16 + \dfrac{48}{x-3}$

$\dfrac{48}{x-3} \to 0$ as $x \to \pm\infty$

So, the graph of f approaches the graph of $y = 2x^2 + 5x + 16$ asymptotically as $x \to \pm\infty$.

Graph is not to scale.... numbers didn't work out for that - but this is the overall layout.



Ex 11 cont'd: Find the slant asymptote for $f(x)=\dfrac{8x^2-7x+1}{2x-3}$.

$\frac{8x^2}{2x} \to 4x \qquad \frac{5x}{2x} \to \frac{5}{2}$

$$
\begin{array}{r|l}
 & 4x + \frac{5}{2} \\ \hline
2x-3 & 8x^2 - 7x + 1 \\
 & -(8x^2 - 12x) \\ \hline
 & \qquad 5x + 1 \\
 & \qquad -(5x - \frac{15}{2}) \\ \hline
 & \qquad\quad \frac{17}{2}
\end{array}
$$

$1+\frac{15}{2} = \frac{2}{2}+\frac{15}{2}$

Note: in polynomial long division, insert placeholders for missing powers.

Ex: $\dfrac{x^4-5x+1}{x+6}$

$$x+6 \,\overline{)\, x^4 + 0x^3 + 0x^2 - 5x + 1}$$

Thus $f(x) = \dfrac{8x^2-7x+1}{2x-3} = 4x + \dfrac{5}{2} + \dfrac{17/2}{2x-3}$

Note: $\displaystyle\lim_{x\to\infty} \frac{17/2}{2x-3} = 0$

As $x \to \pm\infty$, $f(x) \to 4x + \frac{5}{2} + 0$

$$\boxed{\text{Slant asymptote is } y = 4x + \tfrac{5}{2}.}$$

Ex 10 revisited: Find slant asymptote for $f(x) = \dfrac{x^3+4x^2-11x-7}{x^2+6x+1}$

$$
\begin{array}{r|l}
 & x - 2 \\ \hline
x^2+6x+1 & x^3 + 4x^2 - 11x - 7 \\
 & -(x^3 + 6x^2 + x) \\ \hline
 & \qquad -2x^2 - 12x - 7 \\
 & \qquad -(-2x^2 - 12x - 2) \\ \hline
 & \qquad\qquad\qquad -5
\end{array}
$$

(In general, we expect deg(remainder) to be 1 less than deg(divisor). So, for a quadratic divisor, we expect the remainder to have an x in it. But this time, the x's cancelled out.

This term $\to 0$ as $x \to \pm\infty$

$f(x) = x - 2 - \dfrac{5}{x^2-6x+1}$.

Slant asymptote: $\boxed{y = x - 2}$