4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of f is a function whose derivative is f.

i.e. A function F is an antiderivative of f if $F'(x) = f(x)$.

Example 1: $F(x) = x^3 + 5x + 7$ is an antiderivative of $f(x) = 3x^2 + 5$.

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$G(x) = x^3 + 5x + 8$

$H(x) = x^3 + 5x - \frac{13}{2}$

So we have a whole "family" of antiderivatives of f. Family: $J(x) = x^3 + 5x + C$

> Definition: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

> Theorem: If F is an antiderivative of f on an interval I, then all antiderivatives of f on I will be of the form
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$F(x) = x^3 + 5x + C$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$F(x) = x^6 + \sin x + c$

Integration:

<u>Integration</u> is the process of finding antiderivatives.

> $\int f(x)dx$ is called the *indefinite integral* of *f*.
>
> $\int f(x)dx$ is the family of antiderivatives, or the most general antiderivative of *f*.
>
> This means: $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$.
>
> The *c* is called the *constant of integration*.

Integral sign

“with respect to *x*”

$$\int f(x)dx$$

Integrand

<u>Example 4:</u> Find $\int 3x^2 + 5\, dx$.

$\int (3x^2+5)\, dx = \boxed{x^3+5x+c}$

really should have parentheses for 2 or more terms.

<u>Example 5:</u> Find $\int 6x^5 + \cos x\, dx$.

$\boxed{x^6 + \sin x + c}$

<u>Example 6:</u> Find $\int \sec^2 x\, dx$.

$\int \sec^2 x\, dx = \boxed{\tan x + c}$

Rules for Finding Antiderivatives:

Notation in this table: F is an antiderivative of f, G is an antiderivative of g,

Function	Antiderivative
k	$kx+c$
$kf(x)$	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n for $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc x \cot x$	$-\csc x$
$\csc^2 x$	$-\cot x$

1. $\int k \; dx = kx + c$ (k a constant)

2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ ($n \neq -1$)

3. $\int k \; f(x) \; dx = k \int f(x) \; dx$ (k a constant)

4. $\int [f(x)+g(x)] dx = \int f(x) dx + \int g(x) dx$

5. $\int \cos x \, dx = \sin x + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \sec x \tan x \, dx = \sec x + c$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\Rightarrow \int \csc x \cot x \, dx = -\csc x + c$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\Rightarrow \int \csc^2 x \, dx = -\cot x + c$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow \frac{d}{dx}(-\cos x) = \sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

Example 7: Find the general antiderivative of $f(x) = \frac{1}{2}$.

$F(x) = \int f'(x) \, dx = \int \frac{1}{2} \, dx = \boxed{\frac{1}{2}x + c}$

Example 8: Find $\int x^3\,dx$.

$$\int x^3\,dx = \boxed{\frac{x^4}{4}+C}$$

Example 9: Find $\int 7x^2\,dx$.

$$\int 7x^2\,dx = \frac{7x^3}{3}+C = \boxed{\frac{7}{3}x^3+C}$$

Check:

$$\frac{d}{dx}\left(\frac{7}{3}x^3+C\right) = \frac{7}{3}(3x^2)+0 = 7x^2 \checkmark$$

Ex $9\frac{1}{2}$: $\int dz$

$$= \int 1\,dz = \boxed{z+C}$$

Check:

$$\frac{d}{dz}(z+C) = 1+0 = 1 \checkmark \text{ ok}$$

Example 10: Find $\int \frac{1}{x^5}\,dx$.

$$\int x^{-5}\,dx = \frac{x^{-4}}{-4}+C = \boxed{-\frac{1}{4x^4}+C}$$

$$\frac{d}{dx}\left(-\frac{1}{4}x^{-4}\right) = -\frac{1}{4}(-4x^{-5}) = x^{-5} = \frac{1}{x^5} \checkmark$$

Example 11: Find the general antiderivative of $f(x)=\frac{5}{x^2}$.

$$f(x) = 5x^{-2}$$

$$F(x) = \int 5x^{-2}\,dx = \frac{5x^{-1}}{-1}+C = \boxed{-\frac{5}{x}+C}$$

Example 12: $\int\left(6x^2-3x+9\right)dx$

$$\int(6x^2-3x+9)\,dx = 6\frac{x^3}{3}-\frac{3x^2}{2}+9x+C$$

$$= \boxed{2x^3-\frac{3}{2}x^2+9x+C}$$

Check

$$\frac{d}{dx}\left(2x^3-\frac{3}{2}x^2+9x\right) = 6x^2-\frac{3}{2}(2x)+9 = 6x^2-3x+9 \checkmark$$

Ex $12\frac{1}{2}$: $\int \frac{y^4+\sqrt{y}-y}{y^2}\,dy$

$$= \int y^{-2}(y^4+y^{1/2}-y)\,dy$$

$$= \int (y^2+y^{-3/2}-y^{-1})\,dy$$

$$= \frac{y^3}{3}+\frac{y^{-1/2}}{-1/2}-\ln|y|+C$$

$$= \boxed{\frac{1}{3}y^3-2y^{-1/2}-\ln|y|+C}$$

Check:

$$\frac{1}{3}(3y^2)-2\left(-\frac{1}{2}y^{-3/2}\right)-\frac{1}{y}$$

$$= y^2+y^{-3/2}-\frac{1}{y} \checkmark$$

Example 13: Find $\int 3\sqrt{x}\,dx$.

$$\int 3x^{1/2}\,dx = 3\int x^{1/2}\,dx$$

$$= 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C = \frac{3x^{3/2}}{3/2}+C$$

$$= 3\left(\frac{2}{3}\right)x^{3/2}+C = 2x^{3/2}+C = \boxed{2\sqrt{x^3}+C}$$

Example 14: Find $\int (3\cos x + 5\sin x)\, dx$.

$$\boxed{3\sin x - 5\cos x + c}$$

Check it: $\frac{d}{dx}(3\sin x - 5\cos x) = 3\cos x - 5(-\sin x) = 3\cos x + 5\sin x$ ✓

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\, dx$.

$$\int \frac{x^7 - x^{1/3} + 3x^2}{x^5}\, dx = \int \left[\frac{x^7}{x^5} - \frac{x^{1/3}}{x^5} + \frac{3x^2}{x^5}\right] dx = \int \left(x^2 - x^{-14/3} + 3x^{-3}\right) dx$$

$$= \frac{x^3}{3} - \frac{x^{-11/3}}{-11/3} + \frac{3x^{-2}}{-2} + c = \boxed{\frac{1}{3}x^3 + \frac{3}{11\, x^{11/3}} - \frac{3}{2x^2} + c}$$

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin\theta}{3}$.

$$F(\theta) = \int \frac{1}{3}\sin\theta\, d\theta = \frac{1}{3}(-\cos\theta) + c = \boxed{-\frac{1}{3}\cos\theta + c}$$

Check: $\frac{d}{d\theta}\left(-\frac{1}{3}\cos\theta\right) = -\frac{1}{3}(-\sin\theta) = \frac{1}{3}\sin\theta$ ✓ ok

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}}\right) dx$

$$\int \left(x^{1/3} + 2x^{-1/2}\right) dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{4/3}}{4/3} + \frac{2x^{1/2}}{1/2} + c = \boxed{\frac{3}{4}x^{4/3} + 4x^{1/2} + c} = \frac{3\sqrt[3]{x^4}}{4} + 4\sqrt{x} + c$$

Check: $\frac{d}{dx}\left(\frac{3}{4}x^{4/3} + 4x^{1/2}\right) + c$
$= \frac{3}{4}\cdot\frac{4}{3}x^{1/3} + 4\cdot\frac{1}{2}x^{-1/2} = x^{1/3} + 2x^{-1/2}$ ✓

Example 18: $\int (6y^2 - 2)(8y + 5)\, dy$

$$\int (6y^2 - 2)(8y+5)\, dy = \int (48y^3 + 30y^2 - 16y - 10)\, dy$$

$$= 48\cdot\frac{y^4}{4} + 30\cdot\frac{y^3}{3} - 16\cdot\frac{y^2}{2} - 10y + c = \boxed{12y^4 + 10y^3 - 8y^2 - 10y + c}$$

Check: $\frac{d}{dy}(12y^4 + 10y^3 - 8y^2 - 10y + c) = 48y^3 + 30y^2 - 16y - 10$ ✓

Differential equations: (DE)

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find *f*. This is an example of a differential equation.

$$f(x) = \int (x^2 - 7)\,dx = \boxed{\frac{x^3}{3} - 7x + C}$$

(General solution to the DE.

A specific solution would have a fixed value for C.

Check it!

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

$$f(x) = \int f'(x)\,dx = \int (3x^2 + 2\cos x)\,dx$$

$$= 3\frac{x^3}{3} + 2\sin x + C$$

$$= x^3 + 2\sin x + C$$ general solution

We solve for C.

$f(0) = 3 \Rightarrow \quad 3 = f(0) = 0^3 + 2\sin 0 + C.$

$$3 = 0 + 0 + C$$

$$C = 3$$

Specific solution: $\boxed{f(x) = x^3 + 2\sin(x) + 3}$

Check: $f'(x) = 3x^2 + 2\cos x$ ✓ $\quad f(0) = 0^3 + 2\sin(0) + 3 = 3$ ✓

initial conditions

Example 21: Suppose that $f''(x) = 2x^3 - 6x^2 + 6x$, $f'(2) = -1$, and $f(-1) = 4$. Find $f(x)$.

$$f'(x) = \int f''(x)\,dx = 2\frac{x^4}{4} - 6\frac{x^3}{3} + 6\frac{x^2}{2} + C_1$$

$$= \frac{1}{2}x^4 - 2x^3 + 3x^2 + c_1 \quad \text{Need to find } c_1.$$

$$f'(2) = -1 \Rightarrow f'(2) = -1 = \frac{1}{2}(2)^4 - 2(2)^3 + 3(2)^2 + c_1$$

$$-1 = 8 - 16 + 12 + c_1$$

$$-1 = 4 + c_1$$

$$-5 = c_1$$

So, $f'(x) = \frac{1}{2}x^4 - 2x^3 + 3x^2 - 5$

$$f(x) = \int f'(x)\,dx = \frac{1}{2}\cdot\frac{x^5}{5} - 2\cdot\frac{x^4}{4} + 3\cdot\frac{x^3}{3} - 5x + C_2$$

$$= \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + c_2 \quad \text{Need to find } c_2:$$

$$f(-1) = 4 \Rightarrow f(-1) = 4 = \frac{1}{10}(-1)^5 - \frac{1}{2}(-1)^4 + (-1)^3 - 5(-1) + c_2$$

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Example 22: Suppose that $f''(x) = 12x^2 - 18x$, $f(1) = 2$, and $f(-3) = 1$. Find $f(x)$.

(Not done during class)

$$f'(x) = \int f''(x)\,dx = \int (12x^2 - 18x)\,dx = 12\frac{x^3}{3} - 18\frac{x^2}{2} + c_1$$

$$= 4x^3 - 9x^2 + c_1$$

$$f(x) = \int f'(x)\,dx = \int (4x^3 - 9x^2 + c_1)\,dx = \frac{4x^4}{4} - 9\frac{x^3}{3} + c_1 x + c_2$$

$$= x^4 - 3x^3 + c_1 x + c_2$$

$$f(1) = 2 \Rightarrow f(1) = 1^4 - 3(1)^3 + c_1(1) + c_2 = 2$$

$$1 - 3 + c_1 + c_2 = 2$$

$$-2 + c_1 + c_2 = 2$$

$$c_1 + c_2 = 4$$

$$f(-3) = 1 \Rightarrow f(-3) = (-3)^4 - 3(-3)^3 + c_1(-3) + c_2 = 1$$

$$81 + 81 - 3c_1 + c_2 = 1$$

$$162 - 3c_1 + c_2 = 1$$

$$-3c_1 + c_2 = -161$$

System of 2 equations in 2 unknowns
$$\begin{cases} -3c_1 + c_2 = -161 \\ c_1 + c_2 = 4 \end{cases}$$

cont'd next page

Ex 21 cont'd:

$$f(-1) = 4 = \frac{1}{10}(-1)^5 - \frac{1}{2}(-1)^4 + (-1)^3 - 5(-1) + c_2$$

$$4 = -\frac{1}{10} - \frac{1}{2} - 1 + 5 + c_2$$

Multiply by 10: $\quad 40 = -1 - 5 - 10 + 50 + 10c_2$

$$40 = 34 + 10c_2$$

$$6 = 10c_2 \Rightarrow c_2 = \frac{6}{10} = \frac{3}{5}$$

$$\boxed{f(x) = \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + \frac{3}{5}}$$

Check it! $\quad f'(x) = \frac{1}{10}(5x^4) - \frac{1}{2}(4x^3) + 3x^2 - 5$

$$= \frac{1}{2}x^4 - 2x^3 + 3x^2$$

$$f''(x) = \frac{1}{2}(4x^3) - 6x^2 + 6x = 2x^3 - 6x^2 + 6x \checkmark$$

Check the initial conditions too!

Ex 22 cont'd:

Solve the system: $\begin{cases} -3c_1 + c_2 = -161 \\ c_1 + c_2 = 4 \end{cases}$

Subtract: $\quad -4c_1 = -165$

$$c_1 = \frac{-165}{-4} = \frac{165}{4}$$

$$c_1 + c_2 = 4 \Rightarrow \frac{165}{4} + c_2 = 4$$

$$c_2 = \frac{16}{4} - \frac{165}{4} \Rightarrow c_2 = -\frac{149}{4}$$

Solution: $\quad \boxed{f(x) = x^4 - 3x^3 + \frac{165}{4}x - \frac{149}{4}}$ $\quad$ check it!

$$f'(x) = 4x^3 - 9x^2 + \frac{165}{4}$$

$$f''(x) = 12x^2 - 18x \checkmark$$

$$f(1) = 1^4 - 3(1)^3 + \frac{165}{4}(1) - \frac{149}{4}$$

$$= 1 - 3 + \frac{16}{4} = -2 + 4 = 2 \checkmark$$

$$f(-3) = (-3)^4 - 3(-3)^3 + \frac{165}{4}(-3) - \frac{149}{4}$$

$$= 81 + 81 - \frac{495}{4} - \frac{149}{4}$$

$$= 162 - \frac{644}{4} = 162 - 161 = 1$$

$$\begin{array}{r} 165 \\ \times\ 3 \\ \hline 495 \\ +149 \\ \hline 644 \end{array}$$

$$\begin{array}{r} 161 \\ 4\overline{)644} \\ 4 \\ \hline 24 \\ 24 \\ \hline 04 \end{array}$$

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time *t*, then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

Note: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2.

Example 23: Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

Position function: $s(t) = \int s'(t)\,dt = \int v(t)\,dt$

$= \int (2\sin t + \cos t)\,dt$

$= -2\cos t + \sin t + C$

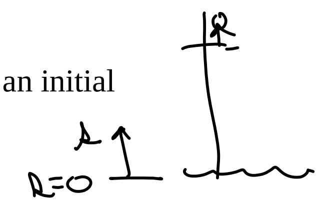
Put in the initial condion: $s(0) = 3 = -2\cos(0) + \sin(0) + C$

$3 = -2(1) + 0 + C$

$5 = C$

Position function: $\boxed{s(t) = -2\cos t + \sin t + 5}$

Example 24: Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?



Acceleration due to gravity: $g = 32$ ft/sec^2
(gravitational constant)

Acceleration function: $s''(t) = a(t) = v'(t) = -32$
(Defining upward to be positive; position = distance above water

Velocity: $s'(t) = v(t) = \int s''(t)\,dt = \int (-32)\,dt = -32t + C$

Initial condition: $v(0) = +40$ ft/sec

$v(0) = s'(0) = 40 = -32(0) + C \Rightarrow C = 40$

So, $v(t) = s'(t) = -32t + 40$

Position: $s(t) = \int s'(t)\,dt = \int (-32t + 40)\,dt = -32 \cdot \frac{t^2}{2} + 40t + C_2$

$= -16t^2 + 40t + C_2$

next page

Initial condition: $s(0) = 30$

$$30 = -16(0)^2 + 40(0) + C_2$$

$$C_2 = 30$$

Position: $\boxed{s(t) = -16t^2 + 40t + 30}$ use this to answer the questions: (see next page)

Vertical Motion due to gravity:

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

where $g =$ gravitational constant (acceleration due to gravity)

$v_0 =$ initial velocity

$s_0 =$ initial position

Back to Ex 24:

How high does it go? Set $v = 0$:

$$v(t) = -32t + 40 = 0$$

$$-32t = -40$$

$$t = \frac{-40}{-32} = \frac{5}{4} = 1.25 \text{ seconds}$$

$$s(1.25) = -16(1.25)^2 + 40(1.25) + 30$$

$$= \boxed{55 \text{ ft}}$$

When does it hit the water?

Set $s(t) = 0$:

$$0 = -16t^2 + 40t + 30$$

Quadratic Formula: $t = \dfrac{-40 \pm \sqrt{40^2 - 4(-16)(30)}}{2(-16)}$

$$= \frac{20 \pm \sqrt{880}}{16} \approx 3.104 \text{ sec}, \cancel{-0.604 \text{ sec}}$$

throw out

It hits the water at $\boxed{3.104 \text{ seconds}}$