4.3: Riemann Sums and Definite Integrals

Riemann sums:

So far we have discussed upper sums, lower sums, right endpoint sums, left endpoint sums, and midpoint sums, all used to approximate the area under the graph of a nonnegative continuous function. These are all examples of *Riemann sums*.

However, the term *Riemann sums* also includes other types of area-related sums. For example, a Riemann sum can use subintervals that vary in width (instead of all subintervals being the same width). The rectangle height can be calculated from any x-value in the subinterval (rather than the right edge, left edge, or midpoint). Also, a Riemann sum can be calculated for functions that take on negative values (in this case, the "height" of the rectangle will be negative, calculated by evaluating the function at that particular x-value).

So for any function, many different Riemann sums can be calculated.

For any Riemann sum: If we allow the maximum subinterval width to approach zero, the number of subintervals will approach infinity, and the Riemann sum will approach the *net area under the curve*, also called the *definite integral* of the function on an interval.

The Definite Integral:

Let f be a continuous function on $[a,b]$. Divide $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$ and let x_i^* be any sample point in the ith interval. Then the *definite integral* of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \text{Net Area}$$

a is called the *lower limit of integration*, and b is called the *upper limit of integration.*

Example 1: Express the limit as a definite integral.

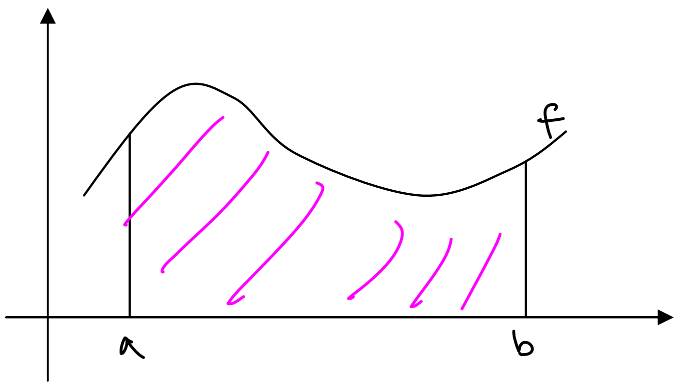
$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - 2\cos x_i)\Delta x,\ [-2\pi, 2\pi]$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - 2\cos(x_i))\Delta x = \int_{-2\pi}^{2\pi}\left(x - 2\cos(x)\right)dx$$

= exact net area under graph of $f(x) = x - 2\cos x$ from -2π to 2π

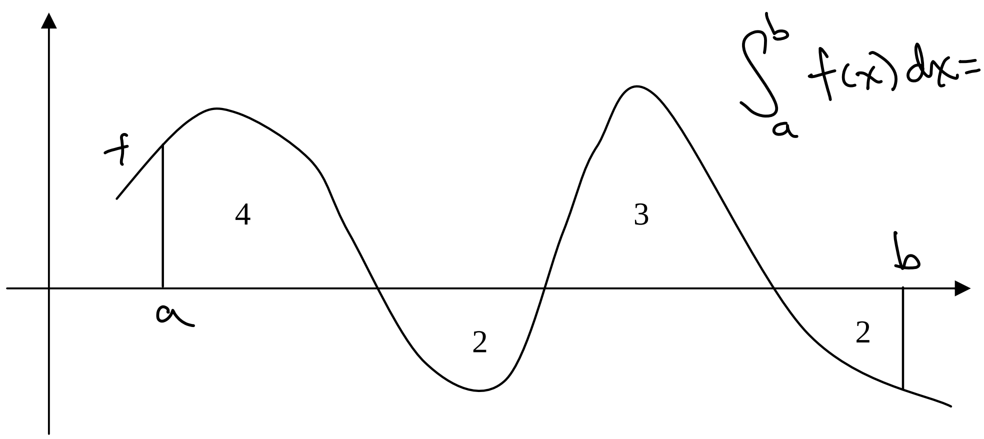
The concept of "net area":

For a function *f* that is nonnegative and continuous on $[a,b]$, the definite integral is the area under the curve from *a* to *b*.

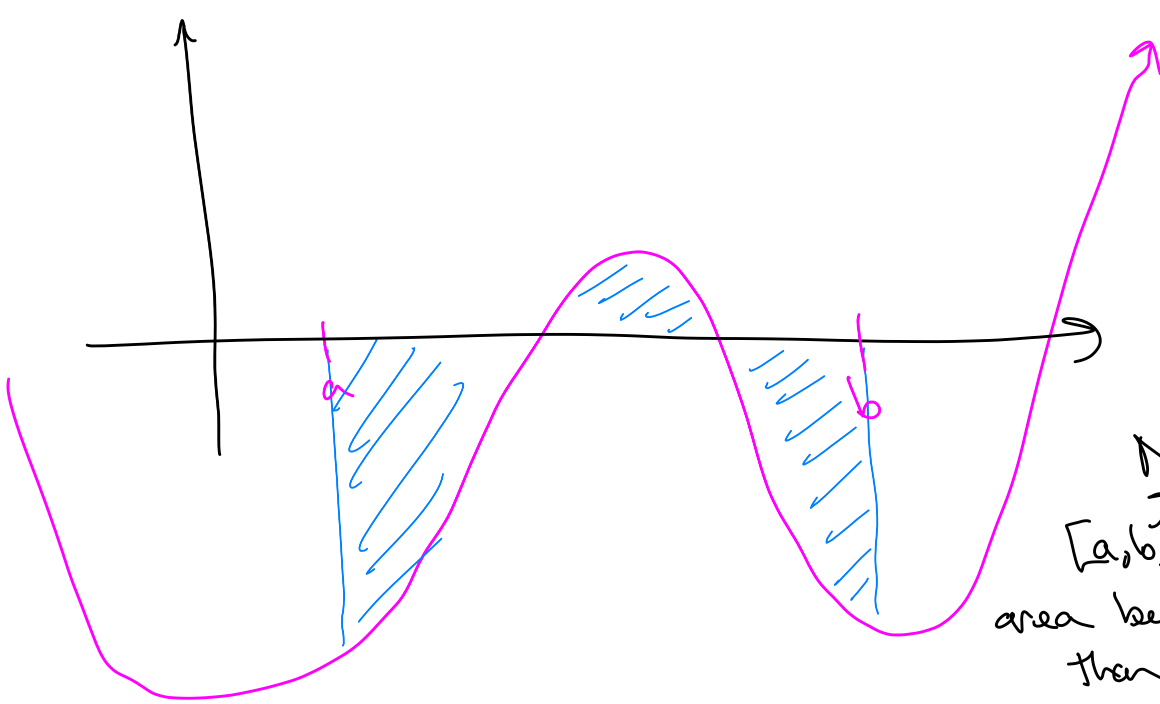


$$\text{Area} = \int_a^b f(x)\,dx$$

For a function *f* that continuous on $[a,b]$ but sometimes negative, the definite integral is the <u>net</u> area under the curve, where areas above the *x*-axis are counted "positively", and areas below the *x*-axis are counted "negatively".



$$\int_a^b f(x)\,dx = 4 - 2 + 3 - 2$$
$$= \boxed{3}$$



Is $\int_a^b f(x)\,dx$ positive or negative?

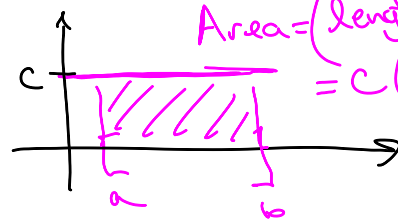
<u>Negative</u>: on $[a,b]$, there is more area below the x-axis than above it.

Properties of Definite Integrals:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b c\,dx = c(b-a)$, where c is any constant.



Area = (length)(width)
$= c(b-a)$

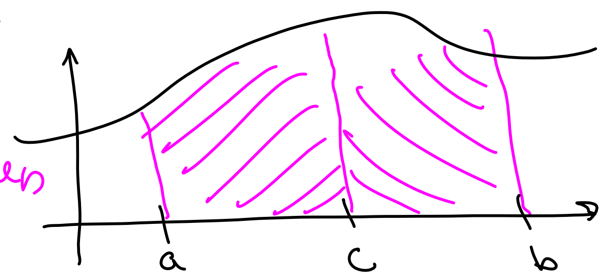
4. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

5. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

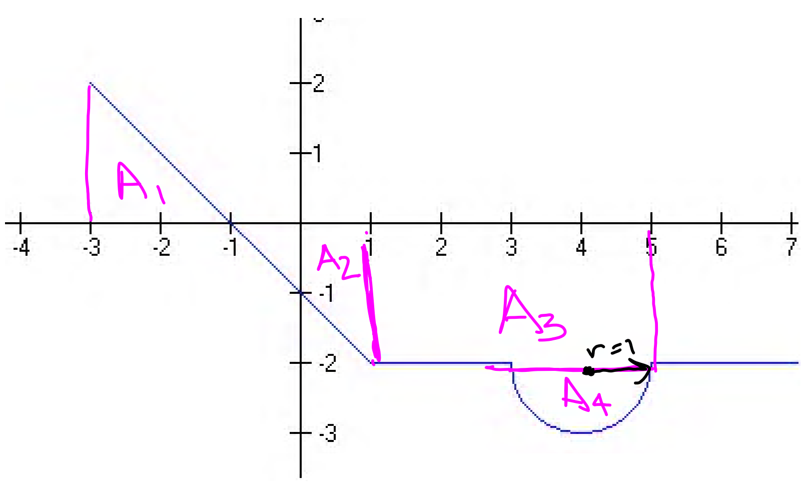
6. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

7. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(still true if c is not in between a and b).



Example 2: For the function below, find $\int_{-3}^{5} f(x)dx$.



$$\int_{-3}^{5} f(x)\,dx = A_1 - A_2 - A_3 - A_4$$

$A_1 = \frac{1}{2}(2)(2) = 2$

$A_2 = \frac{1}{2}(2)(2) = 2$

$A_3 = (2)(4) = 8$

$A_4 = \frac{1}{2}\pi(\text{radius})^2$
$= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

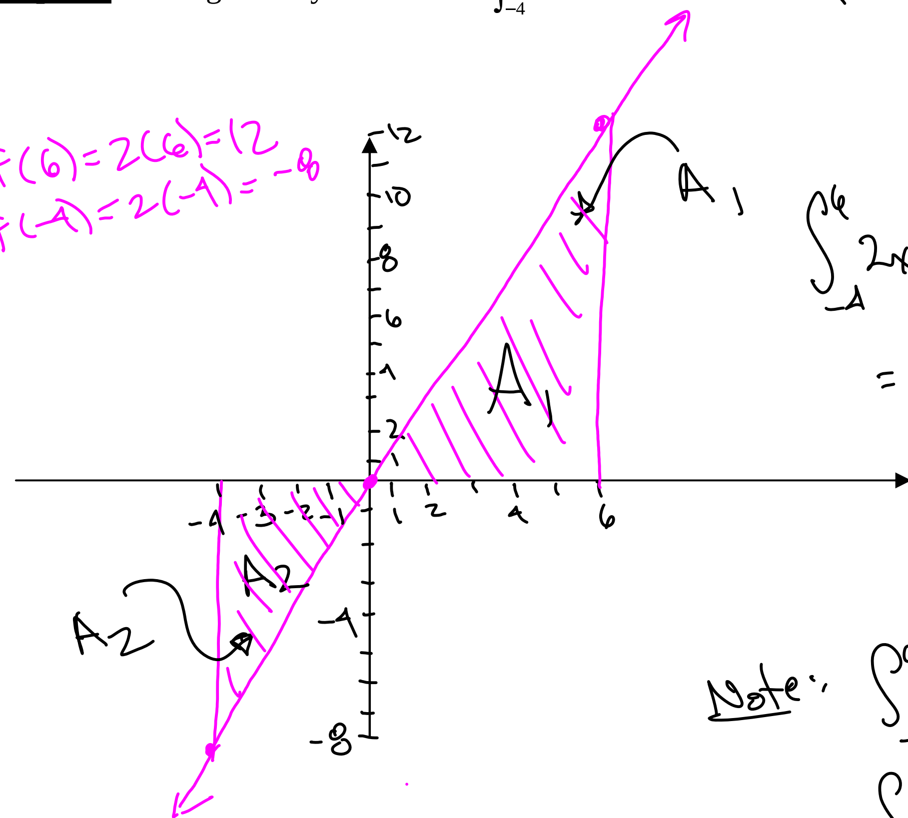
$$\int_{-3}^{5} f(x)\,dx = A_1 - A_2 - A_3 - A_4$$
$$= 2 - 2 - 8 - \frac{\pi}{2}$$
$$= \boxed{-8 - \frac{\pi}{2}}$$

Example 3: Use geometry to calculate $\int_{-4}^{6} 2x \, dx$.

$f(x) = 2x$

Graph it. slope = 2, y-int 0

Note: $f(6) = 2(6) = 12$
$f(-4) = 2(-4) = -8$



$$\int_{-4}^{6} 2x \, dx = A_1 - A_2$$

$$= \frac{1}{2}(6)(12) - \frac{1}{2}(4)(8)$$

$$= 36 - 16$$

$$= \boxed{20}$$

Note: $\int_{-4}^{0} 2x \, dx = -16$

$\int_{0}^{6} 2x \, dx = 36$

(use work from previous example)

~~For the same example~~, calculate $\int_{-4}^{6} (-2x) \, dx$ and $\int_{-4}^{6} 6x \, dx$.

$$\int_{-4}^{6} (-2x) \, dx = - \int_{-4}^{6} 2x \, dx = \boxed{-20}$$

$$\int_{-4}^{6} 6x \, dx = 3 \int_{-4}^{6} 2x \, dx = 3(20) = \boxed{60}$$