3.1: Extrema on an Interval

Absolute maximum and minimum:

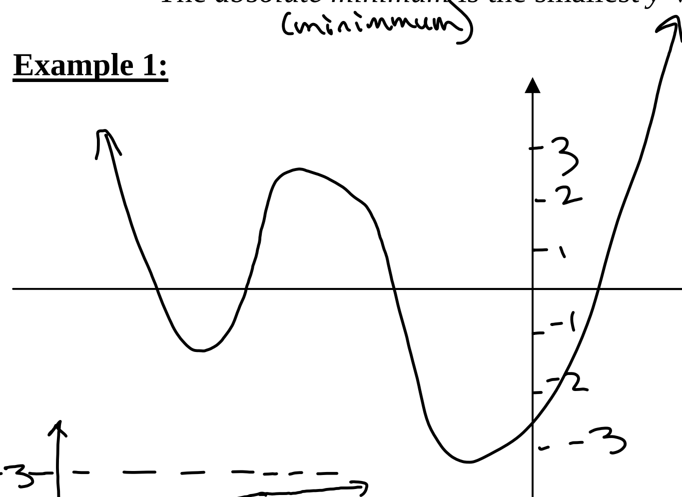
If $f(x) \leq f(c)$ for every *x* in the domain of *f*, then $f(c)$ is the *maximum*, or *absolute maximum*, of *f*.

If $f(x) \geq f(c)$ for every *x* in the domain of *f*, then $f(c)$ is the *minimum*, or *absolute minimum* of *f*.
The maximum and minimum values of a function are called the *extreme values* of the function.

In other words,

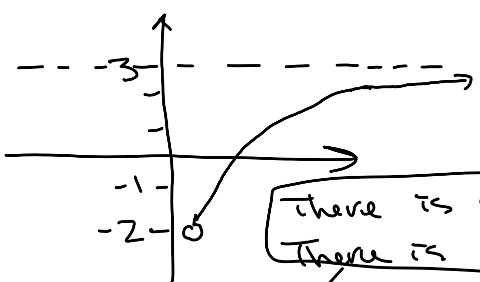
- The *absolute maximum* (maximum) is the largest *y*-value on the graph.
- The *absolute minimum* (minimum) is the smallest *y*-value on the graph.

Example 1:



There is no maximum.
(increases without bound)

The minimum is −3.

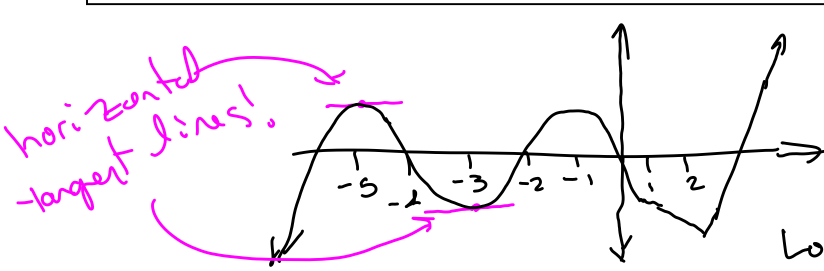


There is no maximum
There is no minimum

Maxima = plural of maximum
Minima = plural of minimum
Extrema: all maxima and minima
Extremum = singula

Relative (Local) Maxima and Minima:

- A function *f* has a *relative maximum*, or *local maximum*, at $x = c$ if there is an interval (a,b) around *c* such that $f(x) \leq f(c)$ for every *x* in (a,b). (These are the "hilltops").
- A function *f* has a *relative minimum*, or *local minimum*, at $x = c$ if there is an interval (a,b) around *c* such that $f(x) \geq f(c)$ for every *x* in (a,b). (These are the "bottoms of valleys").



This function does not have an absolute max or absolute min.
Local (relative) maxima at $x \approx -1, -5$
Local (relative) minima at $x \approx -3, 2$

Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

> Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

This means that if f is differentiable at c and has a relative extreme at c, then the tangent line to f at c must be horizontal.

However, we must be careful. The fact that $f'(c) = 0$ (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at c.

Example 2: $f(x) = x^3$

Where is the tangent line horizontal?

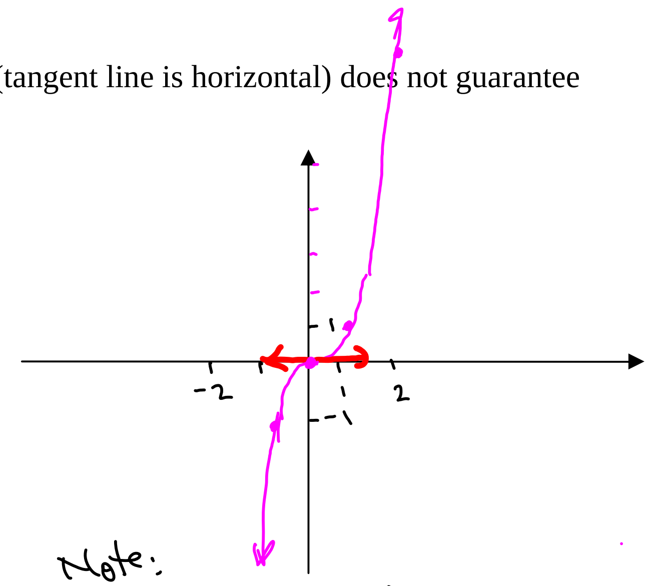
set $f'(x) = 0$:

$f'(x) = 3x^2$

$3x^2 = 0$

$x^2 = 0$

$x = 0$ horizontal tangent at $x=0$



Note:

No relative min

No relative max

Example 3: There can be a local maximum or minimum at c even if $f'(c)$ does not exist.

Note: $f(x) = \tan x$

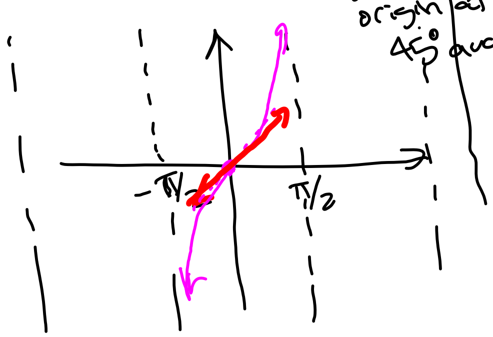
$f'(x) = \sec^2 x$

slope at $x=0$:

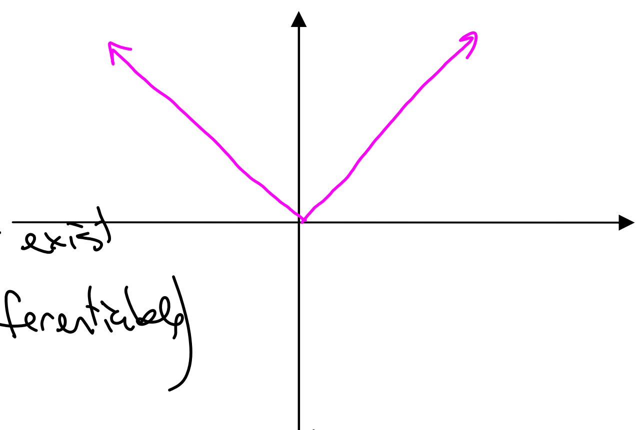
$f'(0) = (\sec(0))^2 = \left(\frac{1}{\cos 0}\right)^2$

$= \left(\frac{1}{1}\right)^2 = 1$

tangent line crosses origin at 45° angle



$f(x) = |x|$



Note:

$f'(0)$ does not exist

(f is not differentiable at 0)

Absolute min at $x=0$

(This is also a local min)

No absolute max.

Critical numbers:

Critical Number: A *critical number* of a function *f* is a number *c* in the domain of *f* such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: If *f* has a local maximum or minimum at *c*, then *c* is a critical number of *f*.

Note: The converse of this theorem is not true. It is possible for *f* to have a critical number at *c*, but not to have a local maximum or minimum at *c*.

Ex: $f(x) = x^3$. 0 is a critical number, but there's not a local max/min at 0.

Example 4: Find the critical numbers of $f(x) = x^3 + \frac{17}{2}x^2 - 6x + 4$.

Set $f'(x)=0$ to find the critical numbers:

$$f'(x) = 3x^2 + \frac{17}{2}(2x) - 6 = 0$$

$$3x^2 + 17x - 6 = 0$$

$$(3x - 1)(x + 6) = 0$$

$3x - 1 = 0$, $3x = 1$, $x = \frac{1}{3}$ | $x + 6 = 0$, $x = -6$

Critical Numbers are $\frac{1}{3}$ and -6.

Example 5: Find the critical numbers of $f(x) = x^{2/3}$

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Where is $f'(x)=0$?
Numerator is never 0, so we can't have $f'(x)=0$.

Graph/limit shows us it has a cusp at 0

where is $f'(x)$ undefined
At $x=0$.
Is $x=0$ in the domain?

Original fcn: $f(x) = x^{2/3} = \sqrt[3]{x^2}$
Defined for all real #s.
So $x=0$ is in the domain; thus it is a critical #.

Critical number: 0

Example 6: Find the critical numbers of $f(x) = \frac{x^2}{x-3}$

$$f'(x) = \frac{(x-3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x-3)}{(x-3)^2}$$

$$= \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} = \frac{x^2 - 6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}$$

Where is $f'(x)=0$?
where the numerator is 0, so at $x=0$, $x=6$.

Where is $f'(x)$ undefined?
undefined where denominator is 0, so at $x=3$.

Note: 3 is not a critical # because it's not in domain

Domain of f: $x \neq 3$

Domain: $(-\infty, 3) \cup (3, \infty)$

Critical numbers 0, 6

Absolute extrema on a closed interval:

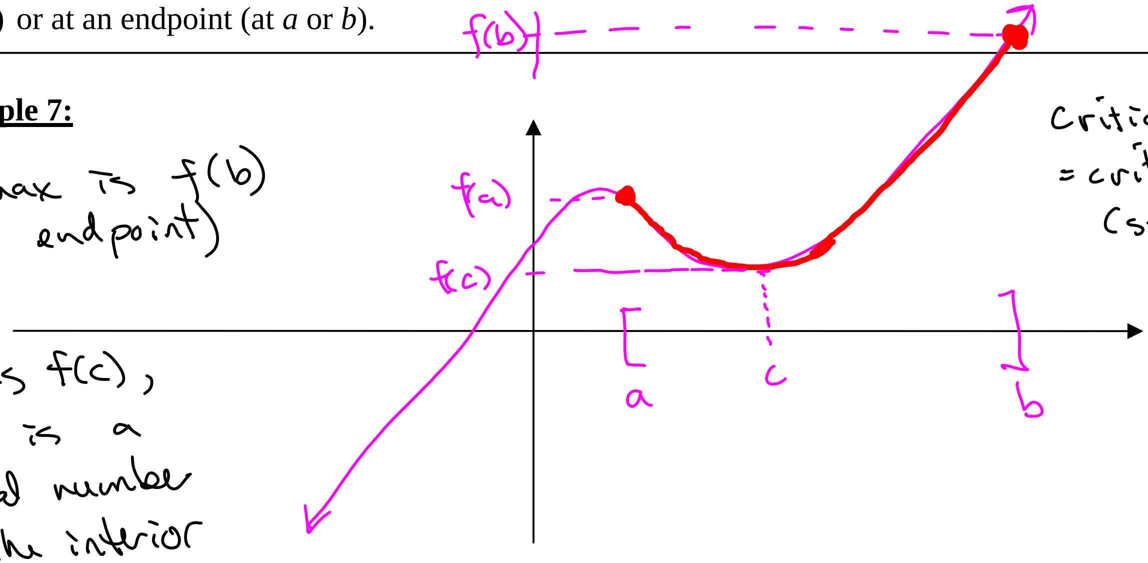
Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

Note: The absolute maximum and the absolute minimum must occur at either a critical value in (a,b) or at an endpoint (at a or b).

Example 7:

Here, the max is $f(b)$ (at an endpoint)

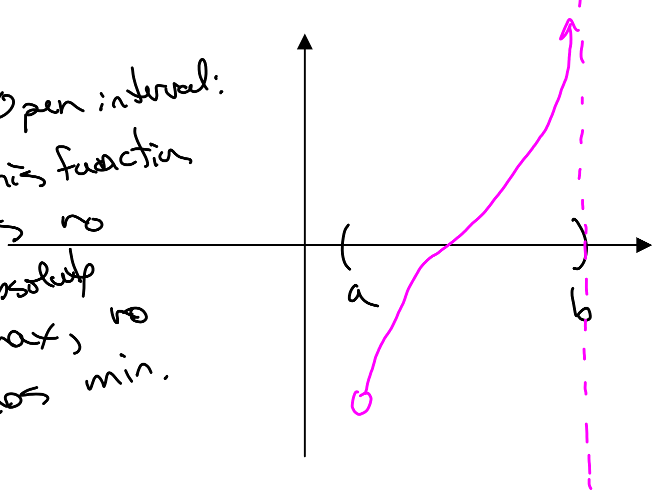
The min is $f(c)$, where c is a critical number in the interior (in (a,b))

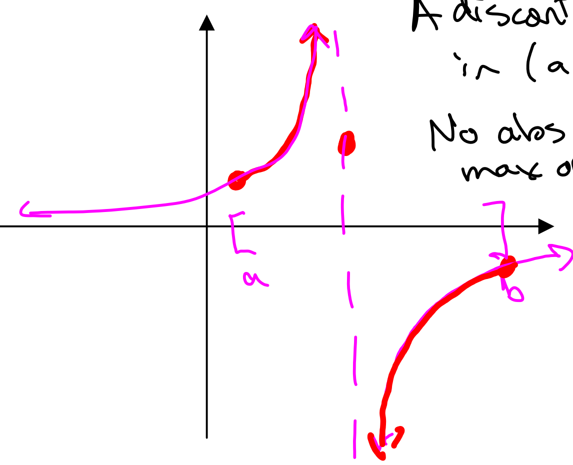


Critical value = critical number (synonyms)

Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.

Open interval: This function has no absolute max, no abs min.





A discontinuity in (a,b)

No abs. max or min.

<u>Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval</u>:

1. Find the critical values in (a,b).
2. Compute the value of f at each critical value in (a,b) and also compute $f(a)$ and $f(b)$.
3. The absolute maximum is the largest of these y-values and the absolute minimum is the smallest of these y-values.

<u>Example 9:</u> Find the absolute extrema for $f(x) = x^2 + 2$ on the interval $[-2,3]$.

If f continuous? Yes

Find critical #s: $f'(x) = 2x = 0$

$x = 0$

Critical number: 0

Is it in $[-2,3]$? Yes

Find y-values for $x = 0, -2, 3$:

$f(0) = 0^2 + 2 = 2$ ← smallest

$f(-2) = (-2)^2 + 2 = 6$

$f(3) = 3^2 + 2 = 11$ ← largest

$\boxed{\text{The absolute max is } f(3) = 11. \quad \text{The absolute min is } f(0) = 2.}$

<u>Example 10:</u> Find the extreme values of $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ on the interval $[-2,1]$.

$g'(x) = \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x$

$= 2x^3 - 2x^2 - 4x$

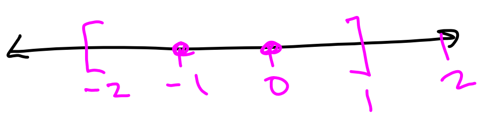
$= 2x(x^2 - x - 2)$

$= 2x(x-2)(x+1)$

$g'(x) = 0$ for $x = 0, 2, -1$

these are the critical #s

which ones are in $[-2,1]$?



Skip $x = 2$, because it's not in our interval.

so find y-values for $-2, -1, 0, 1$:

$g(-2) = \frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - 2(-2)^2 + 3 = \frac{25}{3} = \frac{50}{6}$ ← max

$g(-1) = \frac{1}{2}(-1)^4 - \frac{2}{3}(-1)^3 - 2(-1)^2 + 3 = \frac{13}{6}$

$g(0) = 3$

$g(1) = \frac{1}{2} - \frac{2}{3} - 2 + 3 = \frac{5}{6}$ ← min

$\boxed{\text{The absolute max is } g(-2) = \frac{25}{3} = 8\frac{1}{3} \quad \text{The absolute min is } g(1) = \frac{5}{6}}$

Example 11: Find the absolute extrema of $h(x) = 6x^{2/3}$ on the intervals (a) $[-8,1]$, (b) $[-8,1)$, and. (c) $(-8,1)$.

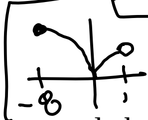
$h(x) = 6x^{2/3} = 6\sqrt[3]{x^2}$ domain $(-\infty,\infty)$. h is continuous on $(-\infty,\infty)$

$h'(x) = 6\left(\frac{2}{3}x^{-1/3}\right) = 4x^{-1/3} = \dfrac{4}{\sqrt[3]{x}}$

Where is $h'(x)=0$? Never... numerator is never 0.
where is $h'(x)$ undefined? at $x=0$. Is this in domain of h?
Yes. So 0 is a critical number.

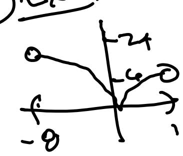
(a) For $[-8,1]$:
$h(-8) = 6\sqrt[3]{(-8)^2} = 6\sqrt[3]{64} = 6(4) = 24$
$h(0) = 6\sqrt[3]{0^2} = 0$
$h(1) = 6\sqrt[3]{1^2} = 6$

Absolute max on $[-8,1]$ is $h(-8)=24$.
Absolute min on $[-8,1]$ is $h(0)=0$.

(b) 

Abs. max on $[-8,1)$ is still $h(-8)=24$
Abs. min on $[-8,1)$ is still $h(0)=0$

Example 12: Find the absolute maximum and absolute minimum of $f(x) = \sin 2x - x$ on the interval $[0,\pi]$.

(c) Ex 11 

There is no absolute max on $(-8,1)$.
The abs. min on $(-8,1)$ is still $h(0)=0$.

f is continuous on $(-\infty,\infty)$.

$f'(x) = (\cos(2x))(2) - 1$
$= -1 + 2\cos 2x$

set $f'(x)=0$: $0 = -1 + 2\cos 2x$
$1 = 2\cos 2x$
$\frac{1}{2} = \cos 2x$

1st think of $\frac{1}{2} = \cos\theta$ [substitute $\theta = 2x$]

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
$\theta = \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$

$\theta = 2x \Rightarrow$ $2x = \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$
$x = \frac{\pi}{6} + k\pi, \frac{5\pi}{6} + k\pi$
k any integer

$0 \le x \le \pi$
$0 \le 2x \le 2\pi$
$0 \le \theta \le 2\pi$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

so $x = \frac{\pi}{6}, \frac{5\pi}{6}$ These are the critical #s in $[0,\pi]$

see next page

Ex 12 cont'd   Find y-values

$f(0) = \sin(2(0)) - 0 = 0 - 0 = 0$

$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.34247$

$f\left(\frac{5\pi}{6}\right) = \sin\left(2\left(\frac{5\pi}{6}\right)\right) - \frac{5\pi}{6} = \sin\left(\frac{5\pi}{3}\right) - \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \approx -3.48402$

$f(\pi) = \sin(2\pi) - \pi = 0 - \pi = -\pi \approx -3.14159$

$$\boxed{\begin{array}{l} \text{Abs. max is } f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \\ \text{Abs. min is } f\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \end{array}}$$