3.2: Rolle's Theorem and the Mean Value Theorem

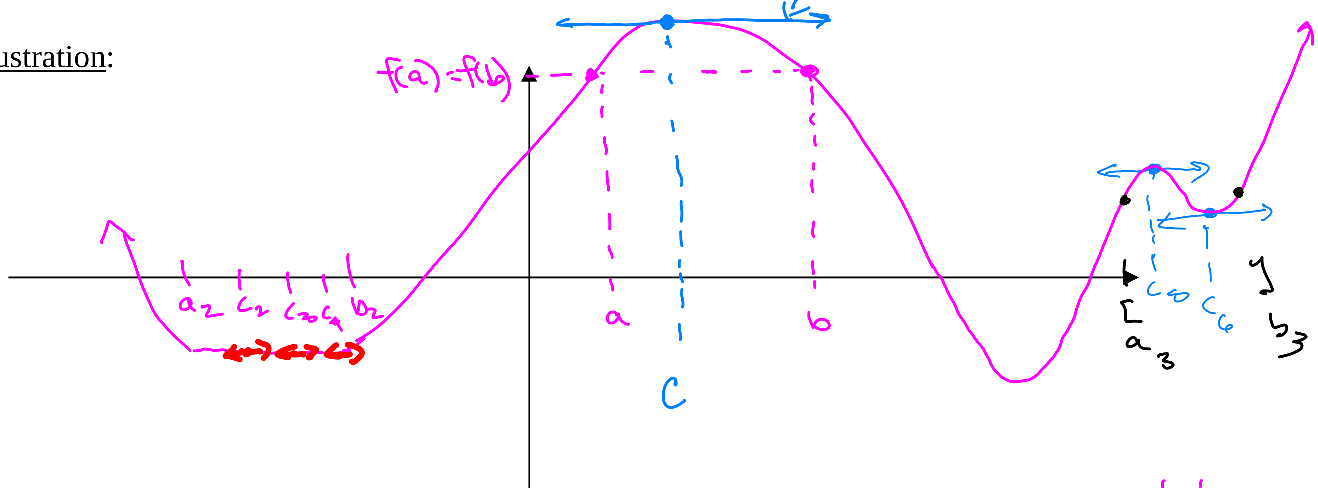
<u>Rolle's Theorem</u>:

Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

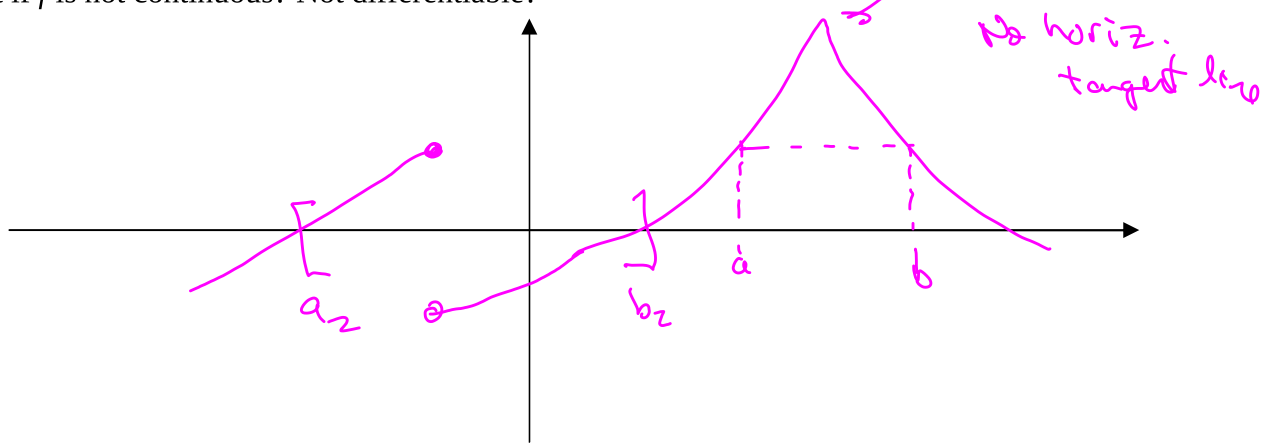
If $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$

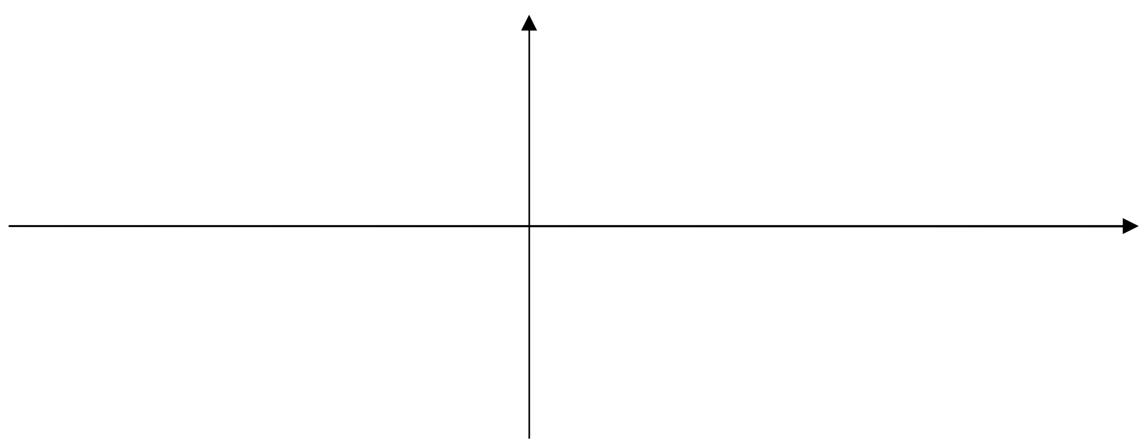
$f'(c)=0$
(horizontal tangent line)

<u>Illustration</u>:



What if f is not continuous? Not differentiable?





Example 1: Show that the function $f(x) = x^2 - 4x - 5$ satisfies the hypotheses of Rolle's Theorem on the interval $[-1,5]$. Find all numbers c in $[-1,5]$ that satisfy the conclusion of Rolle's Theorem.

f is continuous on $(-\infty, \infty)$ and thus on $[-1,5]$.
f is differentiable on $(-\infty,\infty)$ and thus on $(-1,5)$.
(it's a polynomial).

Show $f(-1) = f(5)$:

$f(-1) = (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0$ ✓
$f(5) = 5^2 - 4(5) - 5 = 25 - 25 = 0$ ✓
so $f(-1) = f(5)$.

set $f'(x) = 0$: $f'(x) = 2x - 4 = 0$
$2(x-2) = 0$
$x = 2$

2 is the only c in $[-1,5]$ such that $f'(c) = 0$.

Example 2: Show that the function $g(x) = -2x^4 + 16x^2$ satisfies the hypotheses of Rolle's Theorem on the interval $[-3,3]$. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

g is a polynomial, so it is continuous and differentiable on $(-\infty,\infty)$.

Check that $f(-3) = f(3)$:

$g(3) = -2(3)^4 + 16(3)^2 = g(-3)$.

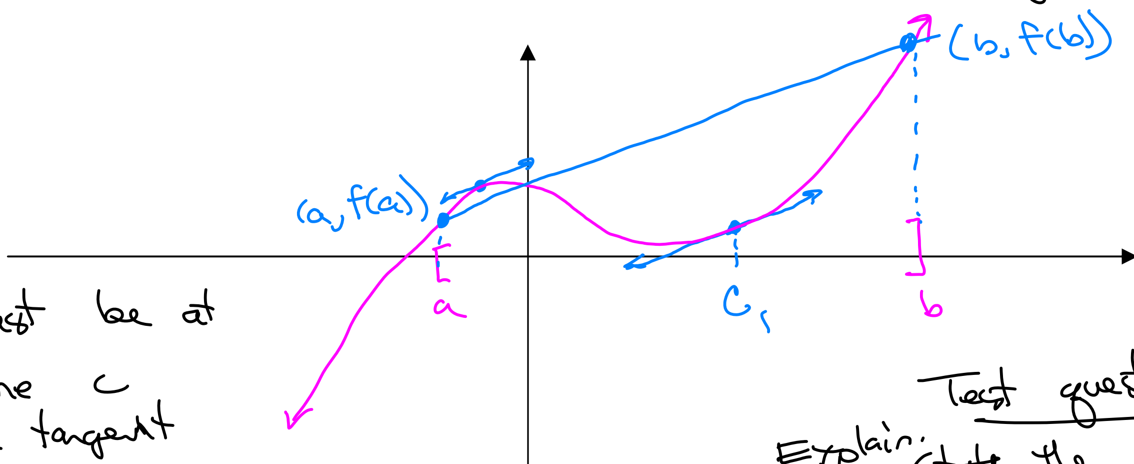
set $g'(x) = 0$:
$g'(x) = -8x^3 + 32x = 0$
$-8x(x^2 - 4) = 0$
$-8x(x+2)(x-2) = 0$
$x = 0, -2, 2$ All are in $[-3,3]$.

There are three c-values that are guaranteed by Rolle's Thm: $0, -2, 2$.

> Mean Value Theorem:
>
> Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).
>
> Then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

(slope of tangent line = slope of line joining the endpoints)

Illustration:



There must be at least one c where the tangent line is parallel to the secant line joining the endpoints.

Test question:
Explain ~~State~~ the mean value theorem and illustrate with a sketch.

A few consequences of the Mean Value Theorem:

1) The Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line that contains the endpoints $(a, f(a))$ and $(b, f(b))$.

2) In terms of rates of change, the Mean Value Theorem guarantees that there is some point at which the <u>instantaneous</u> rate of change is equal to the <u>average</u> rate of change over $[a,b]$.

3) If the derivative of a function is 0 for every number in an interval, then the function is constant on that interval.

4) If two functions have the same derivative on an interval, they differ by a constant on that interval.

Example 3: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(x) = x^{1/2} - 2x$

$f(x) = \sqrt{x} - 2x$ on the interval $[0, 4]$

Domain: $[0, \infty)$

f is continuous on $[0,4]$ and differentiable on $(0,4)$.

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2 = \frac{1}{2\sqrt{x}} - 2$. So derivative exists on $(0,4)$.
($f'(x)$ not defined for $x=0$)

So, it satisfies hypotheses of MVT.

Find slope of secant line connecting endpts:

$f(0) = \sqrt{0} - 2(0) = 0 \Rightarrow$ left endpt is $(0,0)$.

$f(4) = \sqrt{4} - 2(4) = 2 - 8 = -6 \Rightarrow$ right endpt is $(4, -6)$

Slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-0}{4-0} = -\frac{6}{4} = -\frac{3}{2}$

Set $f'(x) = -\frac{3}{2}$:

$f'(x) = \frac{1}{2\sqrt{x}} - 2 = -\frac{3}{2}$

$\frac{1}{2\sqrt{x}} = -\frac{3}{2} + \frac{4}{2}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2}$

$2\sqrt{x} = 2$

$\sqrt{x} = \frac{2}{2} = 1$

$x = 1$

check: $f'(1) = \frac{1}{2\sqrt{1}} - 2 = -\frac{3}{2}$ ✓

So $x=1$ is the only c-value that satisfies the Mean Value Thus

Example 4: As you drive by a Houston police car, your speed is clocked at 50 miles per hour. Five minutes and five miles further, you pass by another HPD officer, who clocks you at 55 miles per hour. If the speed limit on the entire stretch of highway is 55 mph, would the second officer be justified in writing you a speeding ticket?

Slope of secant line = avg rate of change = average velocity in this case (when horizontal axis is time)

Avg Rate of change in distance (avg velocity) $= \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{5 \text{ miles}}{5 \text{ minutes}}$

$= 1$ mile/minute $= 60$ mi/hr

The Mean Value Theorem guarantees that there must be an instant in those 5 minutes when my instantaneous velocity = avg velocity = 60 mph.

(So yes, I deserve a ticket)