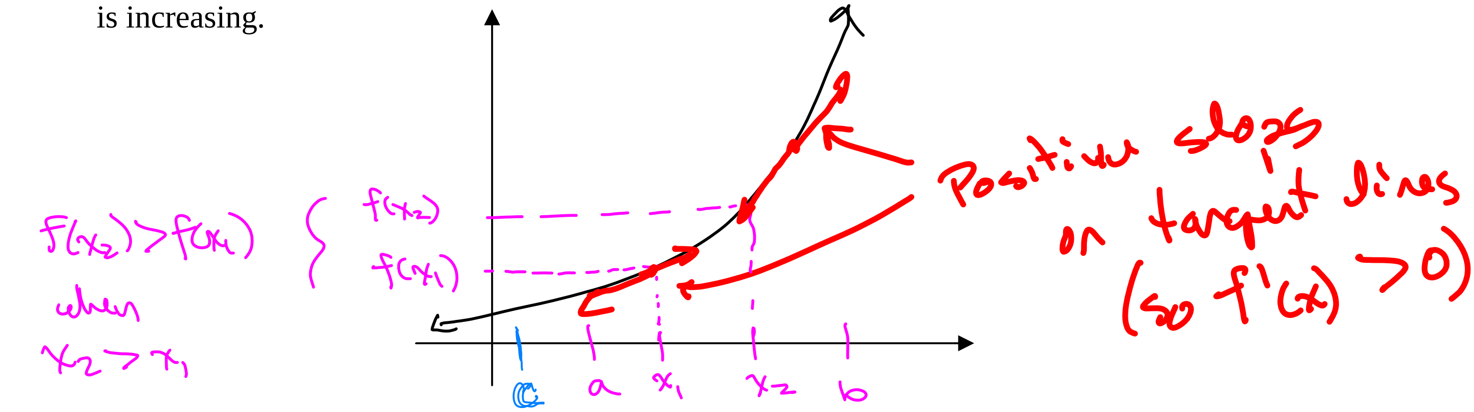
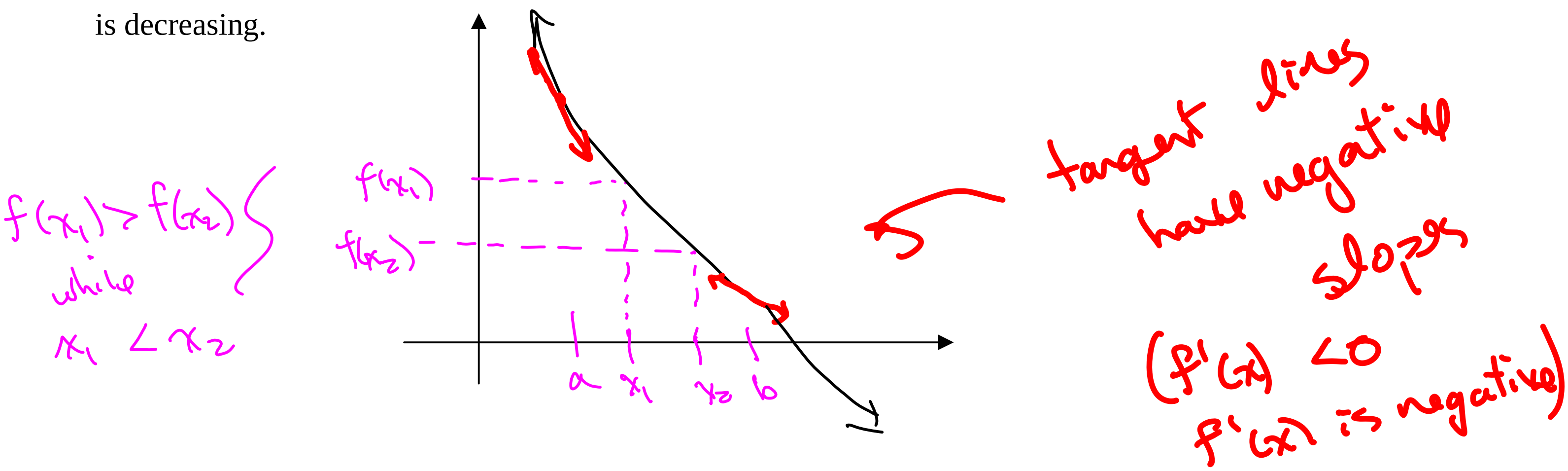
3.3: Increasing and Decreasing Functions and the First Derivative Test

Increasing and decreasing functions:

A function *f* is said to be *increasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. A function *f* is *increasing at c* if there is an interval around *c* on which *f* is increasing.



A function *f* is said to be *decreasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. A function *f* is *decreasing at c* if there is an interval around *c* on which *f* is decreasing.

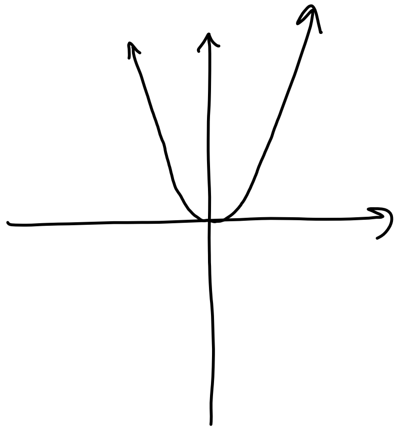


Notice that wherever a function is increasing, the tangent lines have positive slope.
Notice that wherever a function is decreasing, the tangent lines have negative slope.

This means that we can use the derivative to determine the intervals where a function is increasing and decreasing.

> Increasing/Decreasing Test: Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).
> - If $f'(x) > 0$ for every x in (a,b), then *f* is increasing on (a,b).
> - If $f'(x) < 0$ for every x in (a,b), then *f* is decreasing on (a,b).
> - If $f'(x) = 0$ for every x in (a,b), then *f* is constant on (a,b).

Example 1: $f(x) = x^2$



Increasing on $(0, \infty)$.

Decreasing on $(-\infty, 0)$.

Note: $f'(x) = 2x$

For $x > 0$, $f'(x) = 2x > 0 \Rightarrow$ increasing on $(0, \infty)$

For $x < 0$, $f'(x) = 2x < 0 \Rightarrow$ decreasing on $(-\infty, 0)$

Steps for Determining Increasing/Decreasing Intervals

1. Find all the values of x where $f'(x) = 0$ or where $f'(x)$ is not defined. Use these values to split the number line into intervals.

2. Choose a test number c in each interval and determine the sign of $f'(c)$.
   - If $f'(c) > 0$, then f is increasing on that interval.
   - If $f'(c) < 0$, then f is decreasing on that interval.

Note: Three types of numbers can appear on your number line:
1) Numbers where the function is defined and the derivative is 0. (These are critical numbers.)
2) Numbers where the function is defined and the derivative is undefined. (These are also critical numbers.)
3) Numbers where the function is undefined. (These are NOT critical numbers.)

First derivative test:

This procedure determines the relative extrema of a function *f*.

First derivative test:

Suppose that c is a critical number of a function f that is continuous on an open interval containing c.

- If $f'(x)$ changes from positive to negative across c, then f has a relative maximum at c.
- If $f'(x)$ changes from negative to positive across c, then f has a relative minimum at c.
- If $f'(x)$ does not change sign across c, then f does not have a relative extreme at c.

Relative max: $f(-6)=234$
Relative min: $f(2)=-22$

Domain: $(-\infty,\infty)$

Find relative extrema:
Relative max at $x=-6$; rel. min at $x=2$.
Find y-values: $f(-6)=(-6)^3+6(-6)^2-36(-6)+18=234$
$f(2)=(2)^3+6(2)^2-36(2)+18=-22$

Example 2: Determine the intervals on which $f(x)=x^3+6x^2-36x+18$ is increasing and decreasing. Find the relative extrema.

$f'(x)=3x^2+12x-36$
$=3(x^2+4x-12)$
$=3(x+6)(x-2)$.

$f'(x)=0 \Rightarrow x=-6,\ x=2$

Critical numbers: $-6, 2$

f': + + + + + (test -7) | -6 | − − − − (test 0) | 2 | + + + + + + (test 3)

increasing | decreasing | increasing

rel max at $x=-6$

rel. min at $x=2$

$(-\infty,-6)$: Test number $x=-7$:
$f'(-7)=3(-7)^2+12(-7)-36=27$ (+)

or use factored form:
$f'(x)=3(x+6)(x-2)$
$f'(-7)=3(-7+6)(-7-2)=27$
$=3(-1)(-9)$
$(+)(-)(-)\Rightarrow(+)$

$(-6,2)$: Test number $x=0$.
$f'(0)=3(0)^2+12(0)-3=-36$ (−)
or
$f'(0)=3(0+6)(0-2)=3(6)(-2)=-36$
$(+)(+)(-)\Rightarrow(-)$

$(2,\infty)$: Test number $x=3$:
$f'(3)=3(3)^2+12(3)-36=27+36-36=27$ (+)
or $f'(3)=3(3+6)(3-2)\Rightarrow(+)(+)(+)\Rightarrow(+)$

Increasing on $(-\infty,-6)$ and $(2,\infty)$
Decreasing on $(-6,2)$.

Example 3: Determine the intervals on which $g(x)=x^3-6x^2+12x-8$ is increasing and decreasing. Find the relative extrema.

Domain: $(-\infty,\infty)$

$g'(x)=3x^2-12x+12$
$=3(x^2-4x+4)$
$=3(x-2)(x-2)$
$=3(x-2)^2$

Critical number: 2

g': + + + + | 2 | + + + +

increasing | increasing

$(-\infty,2)$: Test # $x=0$
$g'(0)=3(0-2)^2$
$\Rightarrow(+)(-)^2\Rightarrow(+)$

$(2,\infty)$: Test number $x=3$.
$g'(3)=3(3-2)^2\Rightarrow(+)(+)^2$
$\Rightarrow(+)$

Increasing on $(-\infty,\infty)$.
No relative extrema.

$g(x) = x^{2/5} = \sqrt[5]{x^2}$. Domain: $(-\infty, \infty)$

Example 4: Determine the intervals on which $g(x) = x^{2/5}$ is increasing and decreasing. Find the relative extrema.

$g'(x) = \frac{2}{5}x^{-3/5} = \frac{2}{5\sqrt[5]{x^3}}$

where is $g'(x)$ zero? Never (numerator never 0)

where is $g'(x)$ undefined? At $x=0$

Critical number: 0

$$\boxed{\text{g is increasing on } (0,\infty) \text{ and decreasing on } (-\infty,0)}$$

Relative min at $x=0$.

Find y-value: $g(0) = \sqrt[5]{0^2} = 0$

$$\boxed{\text{Relative min is } g(0)=0}$$

g': $- - - - -$ | $+ + + + + +$

decreasing 0 increasing

rel min

$(-\infty, 0)$: Test $x=-1$

$g'(-1) = \frac{2}{5\sqrt[5]{(-1)^3}} = \frac{2}{5(-1)} = -\frac{2}{5}$ $(-)$

$(0,\infty)$: Test number $x=$

$g'(1) = \frac{2}{5\sqrt[5]{(1)^3}} = \frac{2}{5\sqrt[5]{1}} = \frac{2}{5(1)} = \frac{2}{5}$ $(+)$

Example 5: Determine the intervals on which $f(x) = x + \frac{4}{x}$ is increasing and decreasing. Find the relative extrema. Domain: $x \neq 0$

$f(x) = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2}$

$= 1 - \frac{4}{x^2}$

$= \frac{x^2}{x^2} - \frac{4}{x^2}$

$= \frac{x^2-4}{x^2}$

$= \frac{(x+2)(x-2)}{x^2}$

where is f' undefined? $x=0$

(also f undefined at $x=0$)

where is $f'(x)=0$? At $x=2,-2$

(where numerator is 0)

$$\boxed{\text{Critical numbers: } 2, -2}$$

(0 is not a critical # because it is not in domain of f)

rel max rel min

f': $+ + +$ | $- - -$ | $- - -$ | $+ + +$

incr -2 decr 0 decr 2 incr

$(-\infty,-2)$: Test number $x=-3$

$f'(-3) = 1 - \frac{4}{(-3)^2} = 1 - \frac{4}{9} = \frac{5}{9}$ $(+)$

$(-2,0)$: Test $x=-1$

$f'(-1) = 1 - \frac{4}{(-1)^2} = 1 - \frac{4}{1} = -3$ $(-)$

$(0,2)$: Test $x=1$

$f'(1) = 1 - \frac{4}{1} = -3$ $(-)$

$(2,\infty)$: Test $x=3$

$f'(3) = 1 - \frac{4}{3^2} = 1 - \frac{4}{9} = \frac{5}{9}$ $(+)$

Increasing on $(-\infty,-2)$ and on $(2,\infty)$.
Decreasing on $(-2,0)$ and on $(0,2)$.
Relative max. $f(-2) = -2 + \frac{4}{-2} = -4$.
Relative min: $f(2) = 2 + \frac{4}{2} = 4$.

Example 6: Find the local extremes of $g(x) = (x^2 - 4)^{2/3}$. Where is it increasing and decreasing?

See archived notes.

Example 7: Find the relative extremes of $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0, 2\pi)$. Where is it increasing and decreasing on that interval?

See Archives.