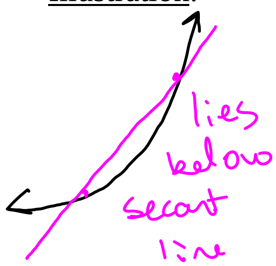
3.4: Concavity and the Second Derivative Test

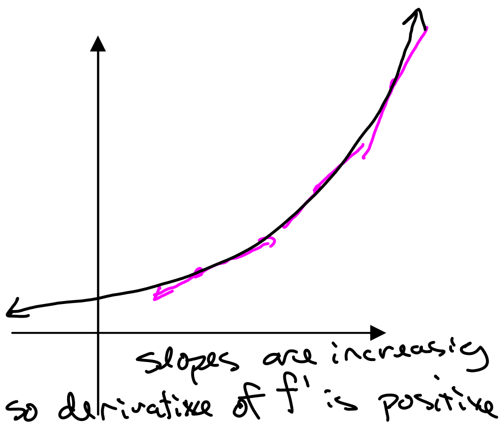
Concavity:

Definition:

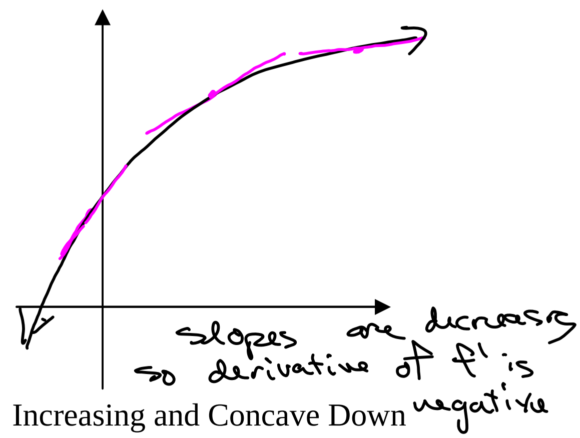
- If the graph of f lies above all of its tangents on an interval, then it is called *concave upward* on that interval.
- If the graph of f lies below all its tangents on an interval, it is called *concave downward* on that interval.

Illustration:

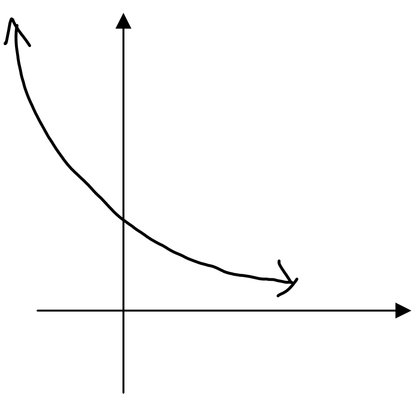




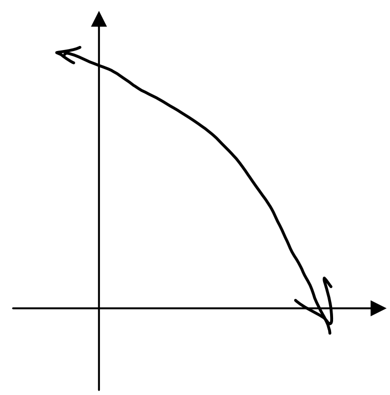
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



Decreasing and Concave Down

Notice the slopes of the tangent lines. When the curve is concave up, the slopes are increasing as you move from left to right.

When the curve is concave down, the slopes are decreasing as you move from left to right.

We find out whether f' is increasing or decreasing by looking at its derivative, which is f''.

<u>Concavity Test</u>:

- If $f''(x) > 0$ for all x in (a,b), then f is <u>concave up</u> on (a,b).
- If $f''(x) < 0$ for all x in (a,b), then f is <u>concave down</u> on (a,b).

<u>Process for Determining Intervals of Concavity</u>:

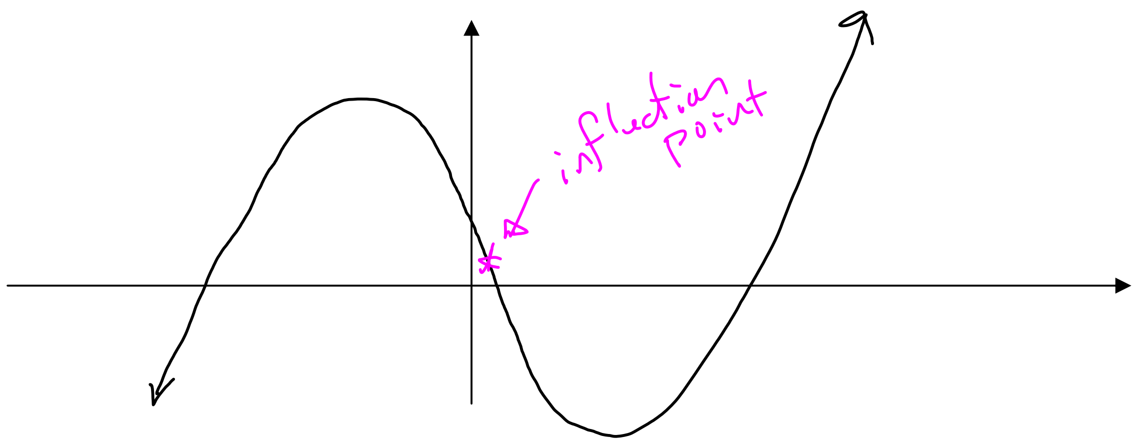
1. Find the values of x where $f''(x) = 0$ or where $f''(x)$ is not defined. Use these values of x to divide the number line into intervals.

2. Choose a test number c in each interval.
   - If $f''(c) > 0$, then f is <u>concave up</u> on that interval.
   - If $f''(c) < 0$, then f is <u>concave down</u> on that interval.

Inflection points:

An *inflection point* is a point on the graph of a function where the concavity changes.

<u>Example 1:</u>



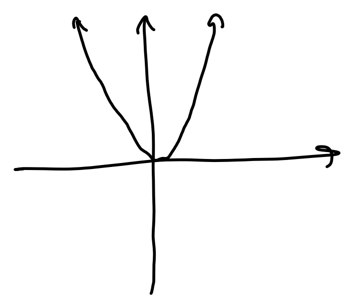
<u>Example 2:</u> Find the intervals on which $f(x) = x^2$ is concave up and concave down.

$f'(x) = 2x$

$f''(x) = 2$

f'' is positive for all x

$\boxed{\text{Concave up on } (-\infty, \infty).}$



Example 3: Determine the intervals of concavity and the inflection points of $f(x) = x^3 + 6x^2 - 36x + 18$.

$f'(x) = 3x^2 + 12x - 36$

$f''(x) = 6x + 12$
$= 6(x+2)$

set $f''(x) = 0$: $0 = 6x + 12$
$-12 = 6x$
$-2 = x$

f'': − − − − − − (concave down) | −2 | + + + + + + (concave up)

$(-\infty, -2)$: Test number $x = -3$

$f''(-3) = 6(-3) + 12 = -18 + 12 = -6$ $(-)$

$(-2, \infty)$: Test number $x = 3$

$f''(3) = 6(3) + 12 = 30$ $(+)$

We have a sign change across -2, so there is an inflection pt. at $x = -2$.

Find y-value: $f(-2) = (-2)^3 + 6(-2)^2 - 36(-2) + 18$
$= -8 + 24 + 72 + 18$
$= 106$

Concave up on $(-2, \infty)$.
Concave down on $(-\infty, 2)$.
Inflection Pt: $(-2, 106)$

Example 4: Determine the intervals of concavity and the inflection points of $f(x) = x + \frac{4}{x}$.

$f(x) = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2}$

$f''(x) = 8x^{-3} = \frac{8}{x^3}$

$f''(x)$ is undefined at $x = 0$. Note that $f(x)$ is also undefined at $x = 0$.

$f''(x)$ is never 0 (because numerator is never 0)

f'': − − − − − (concave down) | 0 | + + + + + + (concave up)

$(-\infty, 0)$: Test $x = -1$:
$f''(-1) = \frac{8}{(-1)^3} = \frac{8}{-1} = -8$ $(-)$

$(0, \infty)$: Test $x = 1$
$f''(1) = \frac{8}{1^3} = 8$ $(+)$

Concave up on $(0, \infty)$.
Concave down on $(-\infty, 0)$.
No inflection points.

The second derivative test:

Notice: For a smooth (differentiable) function, the graph is concave upward at a relative minimum and concave downward at a relative maximum.

Therefore, at a critical number, we can look at the sign of f'' to determine whether there is a relative minimum or relative maximum at that critical number.

> The Second Derivative Test (for Local Extremes):
>
> Suppose f'' is continuous near c.
> - If $f'(x) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.
> - If $f'(x) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.
> - If $f'(x) = 0$ and $f''(c) = 0$, then the test is inconclusive. Use the 1st derivative test instead.

Example 5: Use the second derivative test to find the local extremes of $f(x) = x^3 + 6x^2 - 36x + 18$.

$f'(x) = 3x^2 + 12x - 36$

$= 3(x^2 + 4x - 12)$

$= 3(x+6)(x-2)$

Critical numbers: $-6, 2$ candidates for local max/min

$f''(x) = 6x + 12$

Examine sign of f'' at each critical number:

$f''(-6) = 6(-6) + 12 = -36 + 12 = -24$ $(-)$ $\Rightarrow$ rel max.

$f''(2) = 6(2) + 12$ $(+)$ $\Rightarrow$ rel. min

Relative max $f(-6) = 234$

Relative min $f(2) = -22$

(plug into original fcn to get y-values)

Example 6: Determine the local extremes of $f(x) = -2x^4 + 4x^3$ (using 2nd derivative test)

$f'(x) = -8x^3 + 12x^2$

$= -4x^2(2x - 3)$

$x = 0$ | $2x - 3 = 0$

$2x = 3$

$x = 3/2$

Critical numbers: $0, \frac{3}{2}$

See next page

Ex 6 cont'd

$f'(x) = -8x^3 + 12x^2$

$f''(x) = -24x^2 + 24x$

Examine sign of f'' at critical numbers:

$f''(0) = -24(0)^2 + 24(0) = 0$

2nd derivative test is inconclusive – must use 1st derivative test

$f'(x) = -8x^3 + 12x^2 = -4x^2(2x-3)$

f': + + + + + + + (increasing) | 0 | + + + (incr) | $\frac{3}{2} = 1\frac{1}{2}$ | – – – – – (decreasing)

$(-\infty, 0)$: Test $x = -1$:

$f'(x) = -4x^2(2x-3)$

$\Rightarrow (-)(+)(2x-3)$

$f'(-1) = -4(-1)^2(2(-1)-3)$

$(-)(+)(-2-3)$

$(-)(+)(-)$

$(+)$

$(0, 1\frac{1}{2})$: Test $x = 1$

$(-)(+)(2(1)-3)$

$(-)(+)(-)$

$(+)$

$(1\frac{1}{2}, \infty)$: Test $x = 2$

$(-)(+)(2(2)-3)$

$(-)(+)(+)$

$\Rightarrow (-)$

$$\boxed{\text{Relative max } f\left(\tfrac{3}{2}\right) = \tfrac{27}{8}}$$

Find y value: $f(x) = -2x^4 + 4x^3$

$f\left(\frac{3}{2}\right) = -2\left(\frac{3}{2}\right)^4 + 4\left(\frac{3}{2}\right)^3 = -2\left(\frac{81}{16}\right) + 4\left(\frac{27}{8}\right)$

$= -\frac{81}{8} + \frac{108}{8} = \boxed{\frac{27}{8}} = 3\frac{3}{8}$

Ex 7:

For $f(x) = x^4$, determine the relative extrema.
Use 2nd derivative test if possible.

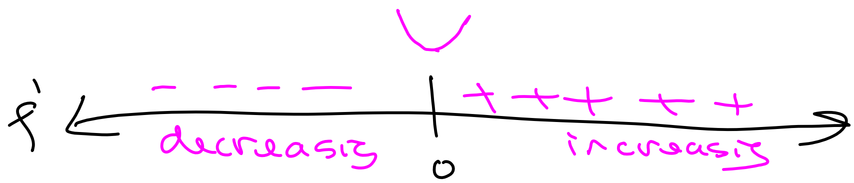
$f'(x) = 4x^3$.

critical number: 0

$4x^3 = 0$
$x^3 = 0$
$x = 0$

$f''(x) = 12x^2$

$f''(0) = 12(0)^2 = 0$ 2nd derivative test is inconclusive



$\boxed{\text{Relative minimum } f(0) = 0}$