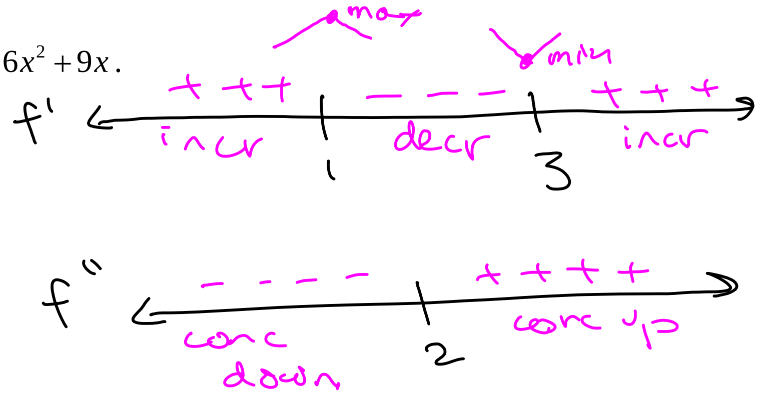
3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of f.
2. Find the x-intercepts and y-intercept, if any.
3. Determine the "end behavior" of f, that is, the behavior for large values of $|x|$ (limits at infinity). (often optional ... can figure out from incr/decreasing intervals)
4. Find the vertical, horizontal, and oblique asymptotes, if any.
5. Determine the intervals where f is increasing/decreasing.
6. Find the relative extremes of f, if any. (You should find both the x- and y-values.)
7. Determine the intervals where f is concave up/concave down.
8. Find the inflection points, if any. (You should find both the x- and y-values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.



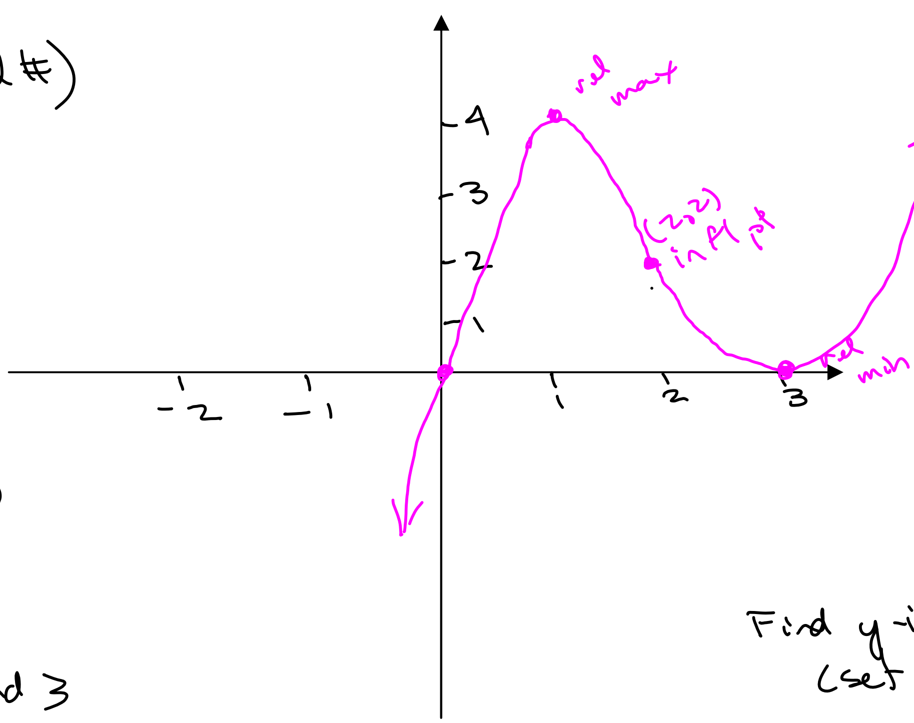
$$f'(x) = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-1)(x-3)$$

Critical numbers: 1, 3

$$f''(x) = 6x - 12$$
$$= 6(x-2)$$

$f''(x) = 0 \Rightarrow x = 2$
(not called a critical #)

Relative max: $f(1) = 4$
Relative min: $f(3) = 0$
Inflection Pt: $(2, 2)$



Find x-intercepts:
$$f(x) = x^3 - 6x^2 + 9x$$
$$= x(x^2 - 6x + 9)$$
$$= x(x-3)^2$$

$y = 0 \Rightarrow x = 0, x = 3$
x-intercepts: 0 and 3
or $(0,0)$ and $(3,0)$

Find y-intercept: 0
(set $x = 0$)

Ex 2: Graph $f(x) = x^3 + 6x^2 - 36x + 18$

(we found incr/decr & concavity intervals in 3.3 and 3.4)

f': + + + | − − − | + + + +

incr (−6) decr (2) incr

f'': − − − − | + + + +

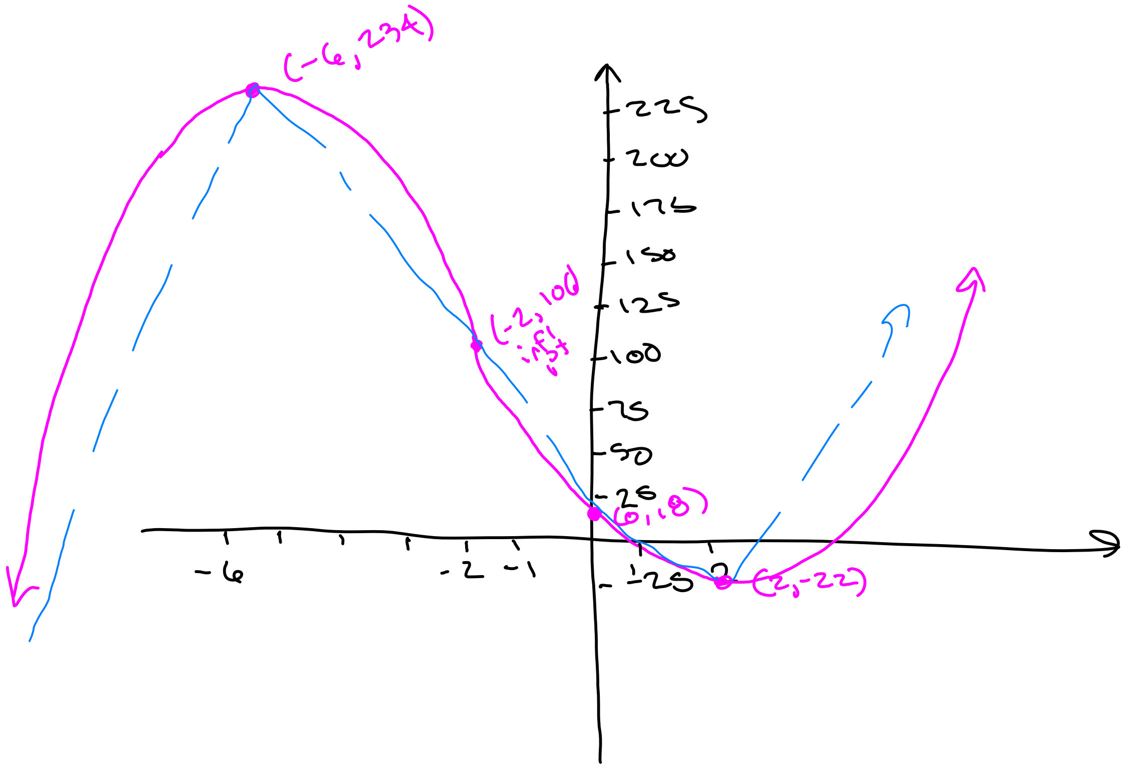
concave down (−2) concave up

rel max $f(-6) = 234$

rel min $f(2) = -22$

Inflection point $(-2, 106)$

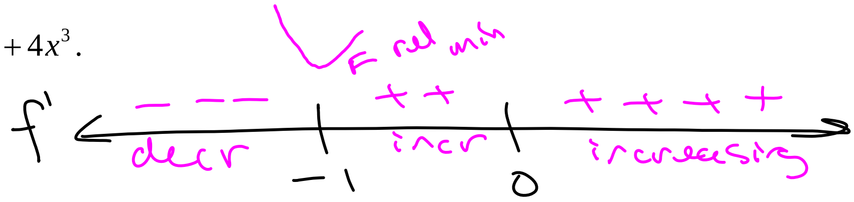
y-intercept: 18 (put in 0 for x)



Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

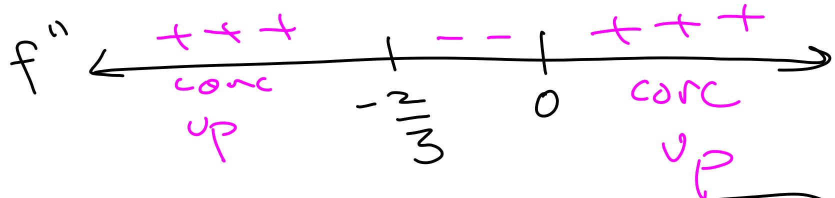
$f'(x) = 12x^3 + 12x^2$
$= 12x^2(x+1)$

Critical numbers: $0, -1$



$f''(x) = 36x^2 + 24x$
$= 12x(3x+2)$

f'' is 0 for $0, -\frac{2}{3}$



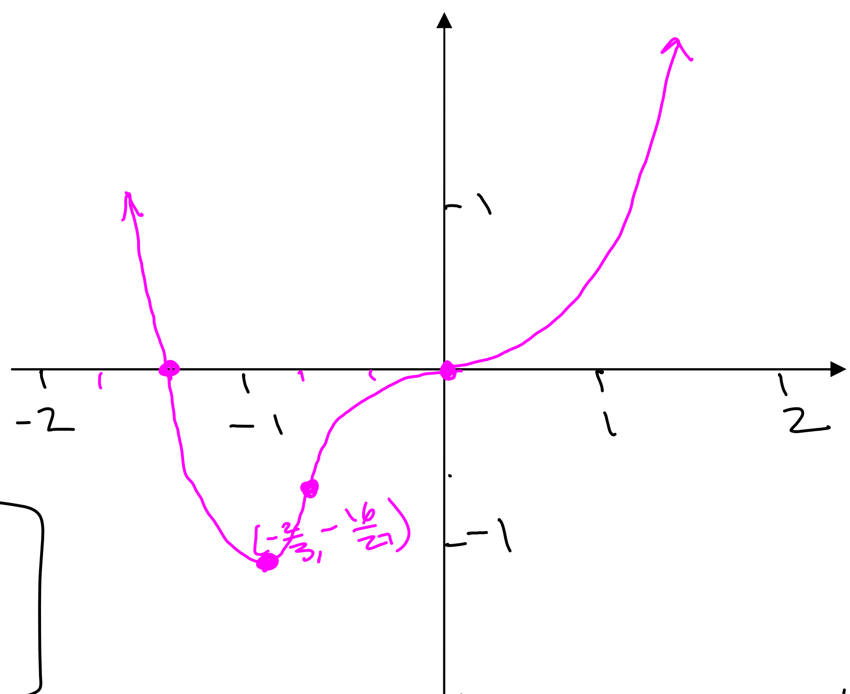
Interval $(-\frac{2}{3}, 0)$: Test $x = -\frac{1}{3}$

$f''(-\frac{1}{3}) = 12(-\frac{1}{3})(3(-\frac{1}{3})+2)$
$= -4(-1+2)$
$= -4(1) = -4$
$(-)$

Increasing on $(-1, \infty)$
Decreasing on $(-\infty, -1)$
Concave up on $(-\infty, -\frac{2}{3}), (0, \infty)$
Concave down on $(-\frac{2}{3}, 0)$

Relative min at $x = -1$.
Find y-value
$f(-1) = 3(-1)^4 + 4(-1)^3$
$= 3 - 4 = -1$

Inflection points at $x = 0$ and $x = -\frac{2}{3}$
Find y-values:
$f(0) = 3(0)^4 + 4(0)^3 = 0$
$f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 = -\frac{16}{27}$



Inflection Pts: $(0,0)$ and $(-\frac{2}{3}, -\frac{16}{27})$

Relative min: $f(-1) = -1$

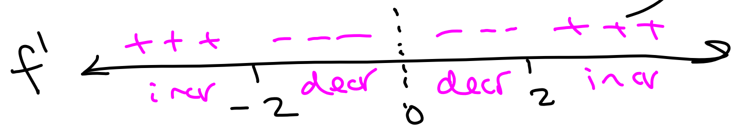
Find x-intercepts: $f(x) = 3x^4 + 4x^3$
$= x^3(3x+4)$
setting $y = 0$
$\Rightarrow x = 0, x = -\frac{4}{3}$

x-intercepts: $0, -4/3$
y-intercept: 0

Find y-intercept: set $x = 0$.
$f(0) = 3(0)^4 + 4(0)^3$
$= 0$

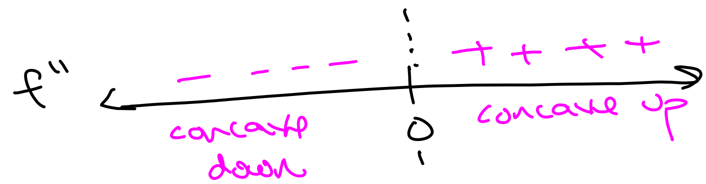
Ex $2\frac{1}{2}$: (Same function as Ex 5 from 3.3 and Ex 4 from 3.4)

$f(x) = x + \frac{4}{x}$



From 3.3 and 3.4

$$\boxed{\begin{array}{l}\text{Relative max: } f(-2) = -4 \\ \text{Relative min: } f(2) = 4 \\ \text{Inflection points: none}\end{array}}$$



$f(x) = \frac{x}{1} + \frac{4}{x}$

$= \frac{x^2+4}{x}$

$\boxed{\text{Vertical asymptote: } x = 0}$

Find y-intercepts: set $x=0$

y is undefined

$\boxed{\text{No y-intercepts}}$

Find x-intercepts:

Set $y=0$: $0 = \frac{x^2+4}{x}$

numerator is never 0, so

$\boxed{\text{no x-intercepts}}$

Find horizontal asymptote:

As $x \to +\infty$,

$\frac{x^2+4}{x} = \frac{\frac{x^2}{x}+\frac{4}{x}}{\frac{x}{x}} = x + \frac{4}{x} \to +\infty$

As $x \to -\infty$,

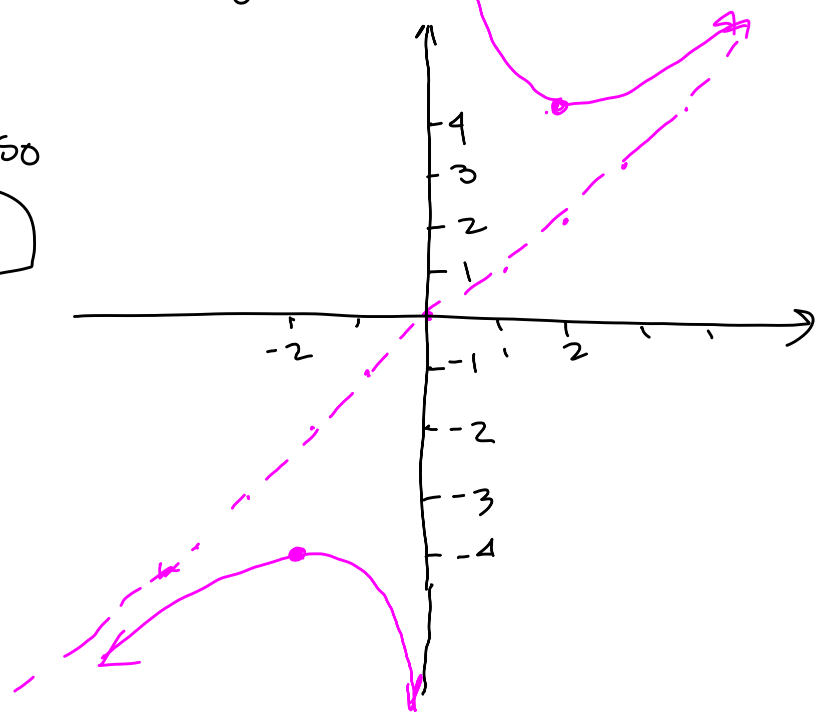
$y = x + \frac{4}{x} \to -\infty$

so it doesn't have a horiz. asymptote

$y = x + \frac{4}{x}$.

The $\frac{4}{x} \to 0$ as $x \to \pm\infty$

Eqn of slant asymptote: $y = x$



Example 3: Sketch the graph of $f(x)=\dfrac{2x}{x^2-1}$.

$$f'(x)=\frac{-2(x^2+1)}{(x^2-1)^2}$$

f'

$$f''(x)=\frac{4x(x^2+3)}{(x^2-1)^3}$$

f''

(Derivative calculations are in archived notes)

From original function: $f(x)=\dfrac{2x}{x^2-1}=\dfrac{2x}{(x+1)(x-1)}$

x-intercept: 0
y-intercept: 0
vertical asymptotes: $x=\pm 1$
horizontal asymptote: $y=0$

Find x-intercepts:
Where is numerator $=0$? At $x=0$
x-intercept: 0

Find y-intercept: set $x=0$:
$y=f(0)=\dfrac{2(0)}{0^2-1}=0$

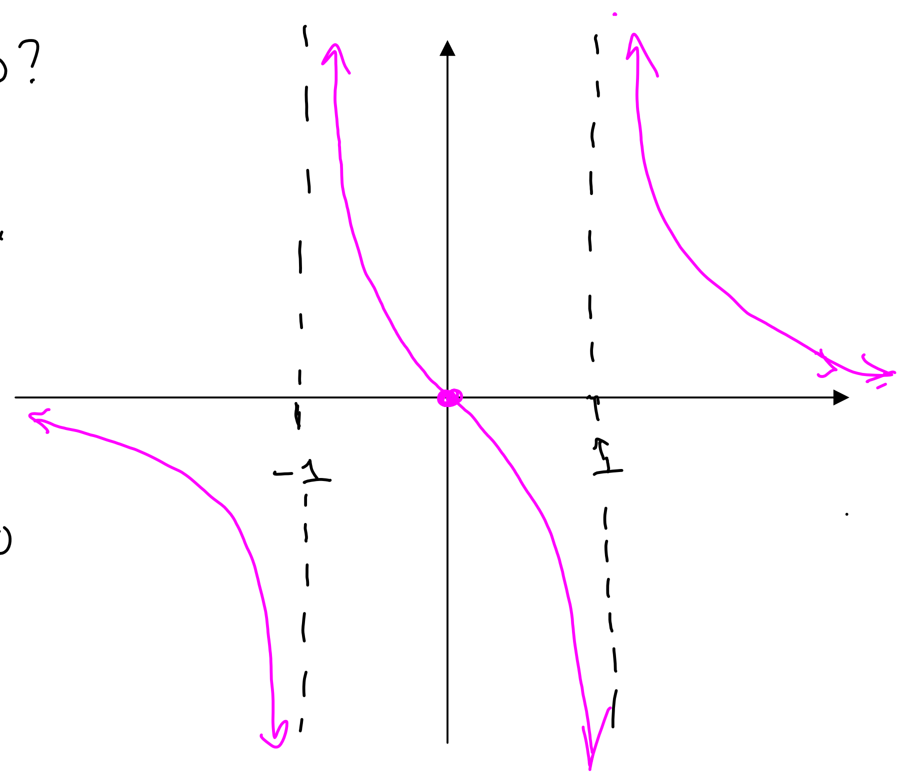
Find vertical asymptotes:
Where is denominator 0?
$x=-1,\ x=1$

Find horizontal asymptotes:

As $x\to\pm\infty$,
$y=\dfrac{2x}{x^2-1}=\dfrac{2/x}{1-\frac{1}{x^2}}$

$\to\dfrac{0}{1-0}=0$

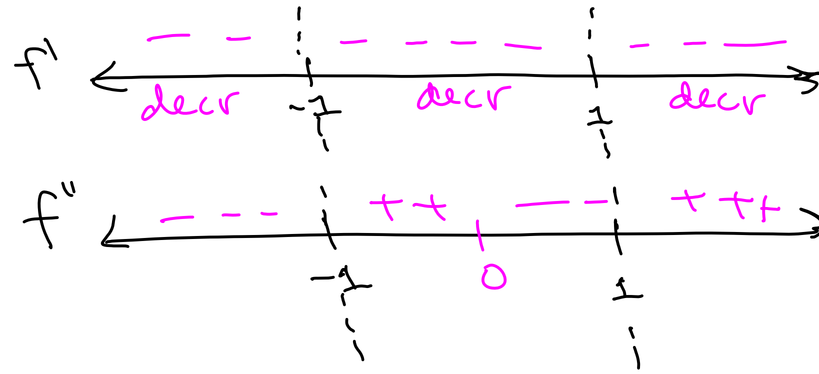
So horiz. asymp. is $y=0$



Ex 3 cont'd

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}$$

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$



From 1st derivative:

Find where $f'(x)=0$: This would mean the numerator is 0.

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}.$$ Numerator is never 0, so $\boxed{\text{no critical numbers.}}$

Where is denominator 0? At $x = \pm 1$. Not in domain of f

In all intervals, $f'(x) \Rightarrow \frac{-2(+)}{(+)} \Rightarrow (-)$

$\boxed{\text{Decreasing on } (-\infty,-1), (-1,1), (1,\infty)}$

$\boxed{\text{No local extrema}}$

From 2nd derivative

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

Where is $f''(x)=0$? $\quad 0 = \frac{4x(x^2+3)}{(x^2-1)^3}$

Where is numerator 0? At $x=0$

Denominator 0 at ± 1, same as original fcn.

$(-\infty,-1)$: Test $x=-2$

$$f''(-2) = \frac{4(-2)(+)}{((-2)^2-1)^3} \Rightarrow \frac{(-)(+)}{(4-1)^3}$$

$$\Rightarrow \frac{(-)}{(+)^3} \Rightarrow (-)$$

$(-1,0)$: Test $x=-0.5$

$$f''(-0.5) \Rightarrow \frac{4(-0.5)(+)}{((-0.5)^2-1)^3} \Rightarrow \frac{(-)(+)}{(0.25-1)^3} \Rightarrow \frac{(-)}{(-)^3} \Rightarrow (+)$$

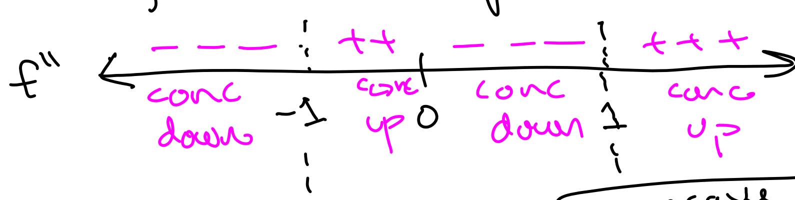
$(0,1)$: Test $x=0.5$

$$f''(0.5) \Rightarrow \frac{4(0.5)(+)}{((0.5)^2-1)^3}$$

$$\Rightarrow \frac{(+)}{(0.25-1)^3} \Rightarrow \frac{(+)}{(-)^3} \Rightarrow (-)$$

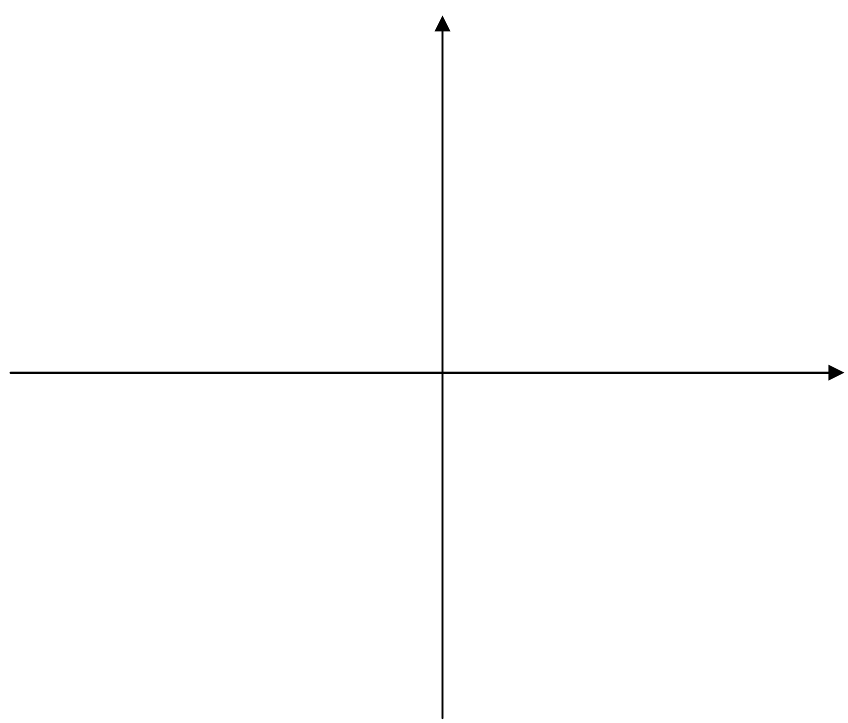
$(1,\infty)$: $(+)$

$\boxed{\text{Inflection Point: } (0,0)}$

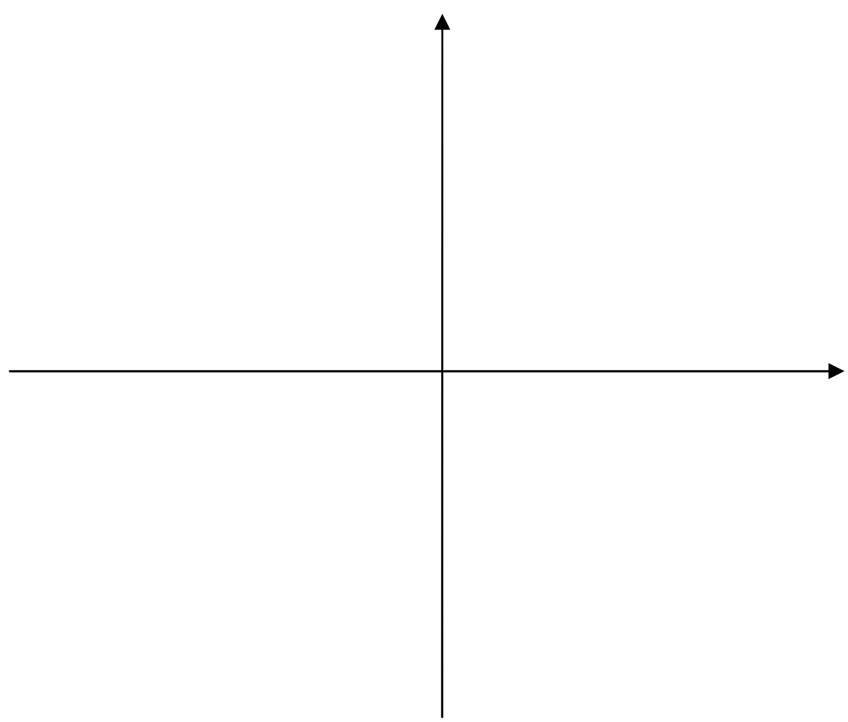


$\boxed{\begin{array}{l}\text{Concave up on } (-1,0) \text{ and } (1,\infty) \\ \text{Concave down on } (-\infty,-1), (0,1)\end{array}}$

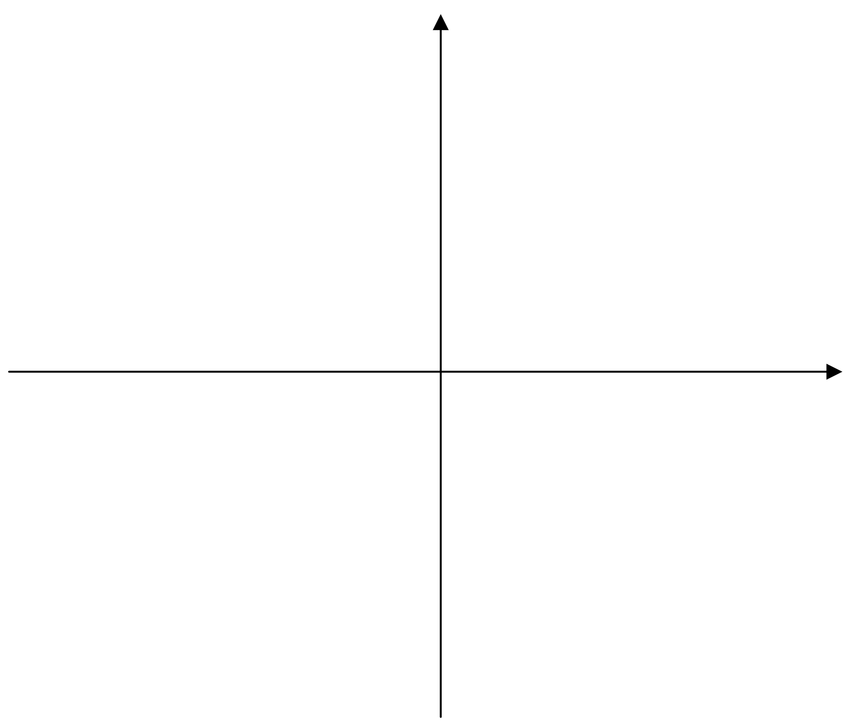
Example 4: Sketch the graph of $f(x)=\dfrac{x^2+1}{x^2-4}$.



Example 5: Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x + 3}$.



Example 6: Sketch the graph of $f(x) = 5x^{2/3} - x^{5/3}$.



Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.