4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as $n \to \infty$ is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

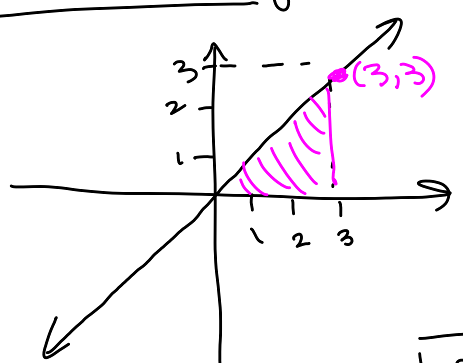
> The Fundamental Theorem of Calculus:
>
> Let f be continuous on the interval $[a,b]$. Then,
>
> $$\overbrace{\int_a^b f(x)dx}^{\text{net area}} = F(b) - F(a)$$
>
> where F is any antiderivative of f; in other words, where $F'(x) = f(x)$.

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of $f(x) = x$ between 0 and 3.

From Geometry:



$$\text{Area} = \frac{1}{2}(3)(3) = \boxed{\frac{9}{2}}$$

Using the Fun Theorem of Calculus

Note: $\int x\,dx = \frac{x^2}{2} + C$ (family of antiderivatives)

$$\text{Area} = \int_0^3 x\,dx = \left(\frac{x^2}{2} + c\right)\Big|_0^3$$

$$= \left(\frac{3^2}{2} + c\right) - \left(\frac{0^2}{2} + c\right)$$

$$= \frac{9}{2} + \cancel{c} - 0 - \cancel{c}$$

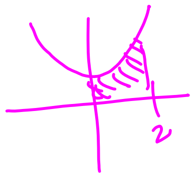
$$= \boxed{\frac{9}{2}}$$

Notice that the constant C disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = F(x)+c\Big|_a^b = \big(F(b)+c\big)-\big(F(a)+c\big) = F(b)+c-F(a)-c = F(b)-F(a)$$

So from now on, we'll omit the "$+c$" when evaluating definite integrals.

Example 2: Find the area under the graph of $f(x)=4x^2+1$ over the interval $[0,2]$. (Compare with our approximation in Section 4.2, Example 5).



$$\text{Area} = \int_0^2 (4x^2+1)\,dx = \left(\frac{4x^3}{3}+x\right)\Big|_0^2$$

$$= \frac{4(2)^3}{3}+2-\left(\frac{4(0)^3}{3}+0\right)$$

$$= \frac{32}{3}+2-0-0 = \frac{32}{3}+\frac{6}{3} = \boxed{\frac{38}{3}}$$

$$= \boxed{12\tfrac{2}{3}} \leftarrow \text{exact area}$$

From Ex 5 in 4.2:

Right endpoints: 17
Left endpts: 9
Midpts: 12.5
} approximations to area

Example 3: Evaluate $\int_{-2}^4 (3x^2-x+4)dx$.

$$\int_{-2}^4 (3x^2-x+4)\,dx = \left(\frac{3x^3}{3}-\frac{x^2}{2}+4x\right)\Big|_{-2}^4 = \left(x^3-\frac{x^2}{2}+4x\right)\Big|_{-2}^4$$

$$= (4)^3-\frac{4^2}{2}+4(4)-\left[(-2)^3-\frac{(-2)^2}{2}+4(-2)\right]$$

$$= 64-8+16-[-8-2-8]$$

$$= 72-[-18] = 72+18 = \boxed{90}$$

Example 4: Evaluate $\int_0^{\pi}(4x^3+\cos x)dx$.

$$\int_0^{\pi}(4x^3+\cos x)\,dx = \left(\frac{4x^4}{4}+\sin x\right)\Big|_0^{\pi}$$
$$= \left(x^4+\sin x\right)\Big|_0^{\pi} = \pi^4+\sin\pi-\left(0^4+\sin 0\right)$$
$$= \pi^4+0-0$$
$$= \boxed{\pi^4}$$

Example 5: Evaluate $\int_1^3\left(\frac{3}{t^2}\right)dt$.

$$\int_1^3\left(\frac{3}{t^2}\right)dt = \int_1^3 3t^{-2}\,dt = \frac{3t^{-1}}{-1}\Big|_1^3 = -\frac{3}{t}\Big|_1^3$$
$$= -\frac{3}{3}-\left(-\frac{3}{1}\right) = -1+3 = \boxed{2}$$

Example 6: Evaluate $\int_2^9\frac{1}{\sqrt{u}}du$.

$$\int_2^9\frac{1}{\sqrt{u}}\,du = \int_2^9 u^{-1/2}\,du = \frac{u^{1/2}}{1/2}\Big|_2^9 = 2\sqrt{u}\Big|_2^9 = 2\sqrt{9}-2\sqrt{2}$$
$$= 2(3)-2\sqrt{2} = \boxed{6-2\sqrt{2}}$$

This is what you would write on the test.

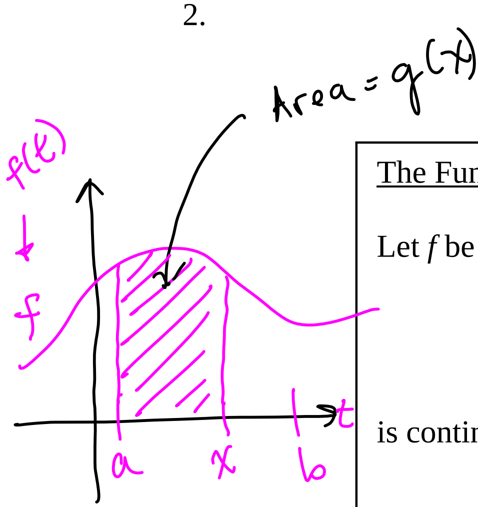
Example 7: Evaluate $\int_{-2}^4\frac{1}{x^3}dx$

$\boxed{\text{Improper Integral}}$

$f(x)=\frac{1}{x^3}$ is not continuous on $[-2,4]$. It is discontinuous at $x=0$. (Re-read the Fun Thm of calculus, carefully!)

So we cannot apply the Fundamental Theorem of Calculus. This is an example of an improper integral. Some improper integrals can be evaluated... we'll learn this in Calculus II.

For now, if f has an infinite discontinuity anywhere in $[a,b]$, assume that $\int_a^b f(x)\,dx$ does not exist. Some of these integrals do exist….you will learn how to handle such integrals in Calculus 2.



<u>The Fundamental Theorem of Calculus, Part II</u>:

Let f be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x)=\int_a^x f(t)\,dt\,, \qquad a\le x\le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)=f(x)$.

In other words, $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$.

Find $\frac{d}{dx}\left(\int_0^x \frac{t^2-2t+4}{t-2}\,dt\right)$

<u>Example 1:</u> Find the derivative of the function $f(x)=\int_3^x \frac{t^2-2t+4}{t-2}\,dt$.

$f(x)$ is the area under the curve of $h(t)=y=\dfrac{t^2-2t+4}{t-2}$

between 3 and x.

Note: $h(t)$ has a discontinuity at $t=2$ but is continuous on $(3,\infty)$.

So, $\boxed{f'(x)=\dfrac{x^2-2x+4}{x-2}}$

<u>Example 2:</u> Find $\frac{d}{dx}\int_{-2}^{\sin x}\sqrt{t^4+2}\,dt$.

$$\frac{d}{dx}\int_{-2}^{\sin x}\sqrt{t^4+2}\,dt$$

$f(t)=\sqrt{t^4+2}$ is continuous on $(-\infty,\infty)$

Area $=\int_{-2}^{\sin x}\sqrt{t^4+2}\,dt$. I want to find $\frac{dA}{dx}$.

Let $u=\sin x$ From the Fun Thm of Calc, Part II,

$$\frac{dA}{du}=\frac{d}{du}\left(\int_{-2}^{u}\sqrt{t^4+2}\right)dt=\sqrt{u^4+2}.$$

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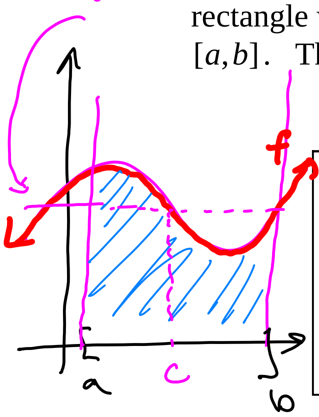
From Chain Rules $\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx}$. We know $\frac{du}{dx} = \frac{d}{dx}(\sin x) = \cos x$.

Then $\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx} = \sqrt{u^4+2}\,(\cos x) = \boxed{\sqrt{(\sin x)^4 + 2}\,(\cos x)}$

The mean (average) value of a function:

$f_{ave} = f(c)$

On the interval $[a,b]$, a continuous function $f(x)$ will have an average "height" c such that the rectangle with width $b-a$ and height c will have the same area as the area under the curve over $[a,b]$. This c is the *average value of the function f over* $[a,b]$.



<u>Mean Value Theorem for Integrals</u>:

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$\underbrace{\int_a^b f(x)\,dx}_{\text{area under curve}} = \underbrace{f(c)}_{\text{height}}\underbrace{(b-a)}_{\text{base}}.$$

This number c is called the *average value* of the function f on the interval $[a,b]$.

The *average value* of a continuous function f on the interval $[a,b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

<u>Example 8:</u> Find the average value of the function $f(x) = 4x^3 - x^2$ over the interval $[-3,2]$.

$$\text{Area} = \int f(x)\,dx = \int_{-3}^{2}(4x^3 - x^2)\,dx = \left(\frac{4x^4}{4} - \frac{x^3}{3}\right)\Big|_{-3}^{2} = \left(x^4 - \frac{x^3}{3}\right)\Big|_{-3}^{2}$$

$$= x^4\Big|_{-3}^{2} - \frac{x^3}{3}\Big|_{-3}^{2} = 2^4 - (-3)^4 - \left[\frac{2^3}{3} - \frac{(-3)^3}{3}\right]$$

$$= 16 - 81 - \left[\frac{8}{3} + 9\right] = -65 - 9 - \frac{8}{3} = -74 - \frac{8}{3} = -\frac{222}{3} - \frac{8}{3} = -\frac{230}{3}$$

$$\text{Area} = -\frac{230}{3} = f_{ave}(b-a)$$

$$f_{ave} = -\frac{230}{3}\left(\frac{1}{2-(-3)}\right) = -\frac{230}{3}\left(\frac{1}{5}\right) = -\frac{230}{15} = -\frac{46}{3}$$

Avg Value is $-\frac{46}{3}$

<u>Example 9:</u> Determine the average value of $f(x) = \sin x$ on the interval $[0,\pi]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}\cos\pi - \left(-\frac{1}{\pi}\cos 0\right)$$

$$= -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1) = \frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}$$