4.5: Integration by Substitution

Most functions cannot be integrated using only the formulas we have learned so far. In Calculus II, you will learn several advanced integration techniques. For now, we'll learn just one new integration technique, called substitution.

Example 1: Find $\int 4(4x-9)^7\,dx$.

One way to do this would be to multiply it out into a long polynomial….YUK!

Here's another way:

Let $u = 4x-9$

$\frac{du}{dx} = 4$

$du = 4\,dx$

$$\int \underbrace{4\,(4x-9)^7\,dx}_{u^7\ du} = \int u^7\,du = \frac{u^8}{8} + C = \boxed{\frac{(4x-9)^8}{8} + C}$$

Check: $\frac{d}{dx}\left(\frac{1}{8}(4x-9)^8\right) = \frac{1}{\cancel{8}}\cdot(\cancel{8})(4x-9)^7(4) = 4(4x-9)^7$ ✓

> The Substitution Rule:
>
> If $u = g(x)$ is a differentiable function whose range is an interval *I* and *f* is continuous on *I*, then
>
> $$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 2: Find $\int \underbrace{6x^2(2x^3+5)^4 dx}_{u^4\ du}$.

$u = 2x^3+5$

$\frac{du}{dx} = 6x^2$

$du = 6x^2\,dx$

$$\int 6x^2(2x^3+5)^4\,dx = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{1}{5}(2x^3+5)^5 + C}$$

check: $\frac{d}{dx}\left(\frac{1}{5}(2x^3+5)^5\right) = \frac{1}{5}(5)(2x^3+5)^4(6x^2) = 6x^2(2x^3+5)^4$ ✓

Example 3: Find $\int -2\sqrt{-2x-4}\;dx$.

$u = -2x-4$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$$\int \underbrace{\left(-2\,(-2x-4)^{1/2}\right)dx}_{u^{1/2}\ du} = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{3}(-2x-4)^{3/2} + C}$$

OR $\int x^3(x^4-1)^6\,dx = \frac{1}{4}\int \underbrace{4x^3(x^4-1)^6\,dx}_{du}$ $\quad u = x^4-1$

$= \frac{1}{4}\int u^6\,du$ ... same as below. $\quad \frac{du}{dx} = 4x^3$, $\quad du = 4x^3dx$

Example 4: Find $\int x^3(x^4-1)^6\,dx$.

$$\int \underbrace{x^3\,dx}_{\frac{1}{4}du}\,\underbrace{(x^4-1)^6}_{u^6} = \int u^6\cdot\frac{1}{4}\,du = \frac{1}{4}\int u^6\,du$$

$$= \frac{1}{4}\left(\frac{u^7}{7}\right) + C = \frac{1}{4}\cdot\frac{(x^4-1)^7}{7} + C$$

$$= \boxed{\frac{1}{28}(x^4-1)^7 + C} = \boxed{\frac{(x^4-1)^7}{28} + C}$$

$u = x^4-1$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

Check: $\frac{d}{dx}\left(\frac{1}{28}(x^4-1)^7\right) = \frac{1}{28}(7)(x^4-1)^6(4x^3) = x^3(x^4-1)^6$ ✓ ok

Example 5: Find $\int \sin 5t\,dt$.

$$\int \underbrace{\sin(5t)}_{\sin(u)}\,\underbrace{dt}_{\frac{1}{5}du} = \frac{1}{5}\int \sin(u)\,du$$

$$= \frac{1}{5}(-\cos(u)) + C = \boxed{-\frac{1}{5}\cos(5t) + C}$$

$u = 5t$

$\frac{du}{dt} = 5$

$du = 5dt$

$\frac{1}{5}du = dt$

Check: $\frac{d}{dt}\left(-\frac{1}{5}\cos(5t)\right) = -\frac{1}{5}(-\sin(5t))(5)$

$= +\sin(5t)$ ✓

Example 6: Find $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

$$\int \underbrace{\frac{1}{\sqrt{x}}\cdot dx}_{2du}\;\underbrace{\cos\sqrt{x}}_{\cos(u)}$$

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$$= 2\int \cos(u)\,du = 2\sin(u) + C$$

$$= \boxed{2\sin(\sqrt{x}) + C}$$

Check: $\frac{d}{dx}\left(2\sin(x^{1/2})\right) = \cancel{2}\cdot(\cos(x^{1/2}))\frac{1}{\cancel{2}}x^{-1/2} = \frac{\cos\sqrt{x}}{\sqrt{x}}$ ✓

Ex $5\frac{1}{2}$: $\displaystyle\int \underbrace{x \, \overbrace{\cos(7x^2)}^{\cos(u)} \, dx}_{\frac{1}{14}du} = \frac{1}{14}\int \cos(u)\,du$

$u = 7x^2$

$\frac{du}{dx} = 14x$

$du = 14x\,dx$

$\frac{1}{14}du = x\,dx$

$= \frac{1}{14}\sin(u) + c = \boxed{\frac{1}{14}\sin(7x^2) + C}$

Check: $\frac{d}{dx}\left[\frac{1}{14}\sin(7x^2) + c\right] = \frac{1}{14}(\cos(7x^2))(14x) = x\cos(7x^2)$ ✓ ok

---

Ex $5\frac{3}{4}$: $\displaystyle\int \sqrt[3]{8x-2}\,dx = \int \underbrace{(8x-2)^{1/3}}_{u^{1/3}} \underbrace{dx}_{\frac{1}{8}du}$

$u = 8x-2$

$\frac{du}{dx} = 8$

$du = 8\,dx$

$\frac{1}{8}du = dx$

$= \frac{1}{8}\int u^{1/3}du = \frac{1}{8}\cdot\frac{u^{4/3}}{4/3} + c = \frac{1}{8}\cdot\frac{3}{4}(8x-2)^{4/3} + c = \boxed{\frac{3}{32}(8x-2)^{4/3} + c}$

Check: $\frac{d}{dx}\left[\frac{3}{32}(8x-2)^{4/3}\right] = \frac{3}{32}\cdot\frac{4}{3}(8x-2)^{1/3}(8) = (8x-2)^{1/3}$ ✓

Ex $5\frac{7}{8}$: $\displaystyle\int \underbrace{\cos(4\theta)}_{\cos(u)} \underbrace{d\theta}_{\frac{1}{4}du}$

$u = 4\theta$

$\frac{du}{d\theta} = 4$

$du = 4\,d\theta$

$\frac{1}{4}du = d\theta$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}\sin(u) + C = \boxed{\frac{1}{4}\sin(4\theta) + c}$

Ex $8\frac{1}{2}$: $I = \displaystyle\int \sin(x)\cos(x)\,dx$

One way:

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$I = \int u\,du$

$= \frac{u^2}{2} + c$

$= \boxed{\frac{\sin^2(x)}{2} + c}$

Another way:

$u = \cos(x)$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$I = \displaystyle\int \underbrace{\sin(x)}_{} \underbrace{\cos(x)}_{u}\,\underbrace{dx}_{}$; $\sin(x)\,dx = -du$

$= -\int u\,du$

$= -\frac{u^2}{2} + c = \boxed{-\frac{\cos^2(x)}{2} + c}$

$\frac{1}{2}(\sin^2 x) + c = \frac{1}{2}(1 - \cos^2 x) + c$

$= \frac{1}{2} - \frac{1}{2}\cos^2 x + c = -\frac{1}{2}\cos^2 x + \underbrace{\frac{1}{2} + c}_{C}$

Example 7: Find $\int \frac{x}{\sqrt[3]{5x^2-8}}\,dx$.

$$\int \underbrace{x\,(5x^2-8)^{-1/3}\,dx}_{\frac{1}{10}du} = \frac{1}{10}\int u^{-1/3}\,du = \frac{1}{10}\cdot\frac{u^{2/3}}{2/3} + c$$

$$= \frac{1}{10}\cdot\frac{3}{2}(5x^2-8)^{2/3} + c = \boxed{\frac{3}{20}(5x^2-8)^{2/3} + c}$$

$u = 5x^2-8$

$\frac{du}{dx} = 10x$

$du = 10x\,dx$

$\frac{1}{10}du = x\,dx!$

Check: $\frac{d}{dx}\left(\frac{3}{20}(5x^2-8)^{2/3}\right) = \frac{3}{20}\left(\frac{2}{3}\right)(5x^2-8)^{-1/3}(10x)$

$= x(5x^2-8)^{-1/3}$ ✓

Example 8: Find $\int \cos x \sin^5 x\,dx$

$$\int \cos(x)\,(\sin(x))^5\,dx = \int u^5\,du$$

$$= \frac{u^6}{6} = \boxed{\frac{1}{6}(\sin(x))^6 + c}$$

$u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)\,dx$

Example 9: Find $\int \sec^2\left(\frac{x}{7}\right)dx$.

$$\int \sec^2\left(\frac{1}{7}x\right)dx = 7\int \sec^2(u)\,du$$

$$= 7\tan(u) + C$$

$$= \boxed{7\tan\left(\frac{1}{7}x\right) + C}$$

Check it!

$u = \frac{x}{7} = \frac{1}{7}x$

$\frac{du}{dx} = \frac{1}{7}$

$du = \frac{1}{7}dx$

$7du = dx$

Example 10: Find $\int x(5+x)^4\,dx$.

$$\int x\,\underbrace{(5+x)^4}_{u^4}\,\underbrace{dx}_{du} = \int x u^4\,du$$

$$= \int (u-5)(u^4)\,du$$

$$= \int (u^5 - 5u^4)\,du = \frac{u^6}{6} - \frac{5u^5}{5} + C$$

$$= \frac{u^6}{6} - u^5 + c = \boxed{\frac{1}{6}(5+x)^6 - (5+x)^5 + c}$$

$u = 5+x$

$\frac{du}{dx} = 1$

$du = dx$

$u - 5 = x$

See next Page

Ex 10 cont'd Check:

$$\frac{d}{dx}\left(\frac{1}{6}(5+x)^6-(5+x)^5\right)=\frac{1}{6}(6)(5+x)^5(1)-5(5+x)^4(1)$$
$$=(5+x)^5-5(5+x)^4=(5+x)^4\left[5+x-5\right]=(5+x)^4(x)\ \checkmark$$

Definite integrals:

<u>The Substitution Rule for Definite Integrals</u>:

If g' is continuous on $[a,b]$ and f is continuous on the range of $u=g(x)$, then

$$\int_a^b f(g(x))g'(x)dx=\int_{g(a)}^{g(b)} f(u)du.$$

There are two methods of evaluating definite integrals when substitution is involved:

<u>Method 1</u>: Find the antiderivative using substitution and switch back to your original variable. Then evaluate the definite integral using the original upper and lower limits.

<u>Method 2</u>: (Using the Substitution Rule for Definite Integrals) Find the antiderivative using substitution, but don't switch back to your original variable. Instead, calculate the upper and lower limits in terms of *u* (or whatever variable you used to substitute). Then evaluate your definite integral using these "new" upper and lower limits.

<u>Example 11:</u> Evaluate $\int_{-1}^{2} x^4(2x^5-8)^3 dx$ using both methods.

$u=2x^5-8$

$\frac{du}{dx}=10x^4$

$du=10x^4dx$

$\frac{1}{10}du=x^4dx$

<u>Method 1</u>: $\int_{-1}^{2}x^4(2x^5-8)^3dx=\frac{1}{10}\int_{x=-1}^{x=2}u^3du$

$$=\frac{1}{10}\cdot\frac{u^4}{4}\Bigg|_{x=-1}^{x=2}=\frac{1}{40}(2x^5-8)^4\Bigg|_{x=-1}^{x=2}=\frac{1}{40}(2(2)^5-8)^4-\frac{1}{40}(2(-1)^5-8)^4$$
$$=\frac{1}{40}(56)^4-\frac{1}{40}(-10)^4=\frac{9824496}{40}=\boxed{245612.4}$$

<u>Method 2</u>: (changing the limits of integration)

$u=2x^5-8$

$\frac{du}{dx}=10x^4$

$du=10x^4dx$

$\frac{1}{10}du=x^4dx$

$x=-1\Rightarrow u=2(-1)^5-8$
$=-2-8=-10$

$x=2\Rightarrow u=2(2)^5-8$
$=64-8=56$

$$\int_{-1}^{2}x^4(2x^5-8)^3dx=\frac{1}{10}\int_{-10=u}^{56}u^3du$$
$$=\frac{1}{10}\cdot\frac{u^4}{4}\Bigg|_{-10}^{56}=\frac{1}{40}\left((56)^4-(-10)^4\right)$$
$$=\frac{1}{40}(9834496-10000)$$
$$=\frac{1}{40}(9824496)=\boxed{245612.4}$$

Example 12: Evaluate $\int_{-4}^{-2} \frac{4}{(3-5x)^3}\,dx$

$u = 3-5x$
$du = -5\,dx$
$-\frac{1}{5}\,du = dx$
$x=-4 \Rightarrow u = 3-5(-4) = 23$
$x=-2 \Rightarrow u = 3-5(-2) = 13$

$$\int_{-4}^{-2} 4(3-5x)^{-3}\,dx = 4\left(-\frac{1}{5}\right)\int_{23}^{13} u^{-3}\,du$$

$$= -\frac{4}{5}\cdot\frac{u^{-2}}{-2}\Bigg|_{23}^{13} = \frac{4}{10}\cdot\frac{1}{u^2}\Bigg|_{23}^{13} = \frac{2}{5}\left(\frac{1}{13^2}-\frac{1}{23^2}\right)$$

$$\approx \boxed{0.0016107} = \boxed{\frac{144}{89401}}$$

Example 13: Evaluate $\int_{\pi/4}^{\pi/3} \frac{\sin x}{\cos^3 x}\,dx$.

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$
$x = \frac{\pi}{4} \Rightarrow u = \cos\frac{\pi}{4}$
$= \frac{\sqrt{2}}{2}$
$x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3}$
$= \frac{1}{2}$

$$\int_{\pi/4}^{\pi/3} \frac{\sin x}{\cos^3 x}\,dx = \int_{\pi/4}^{\pi/3} \underbrace{\sin x\,dx}_{-du}\;\underbrace{(\cos x)^{-3}}_{u^{-3}}$$

$$= -\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} u^{-3}\,du = -\frac{u^{-2}}{-2}\Bigg|_{\sqrt{2}/2}^{1/2} \qquad \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

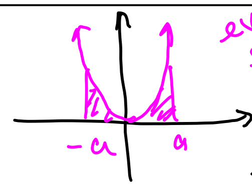
$$= \frac{1}{2u^2}\Bigg|_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} = \frac{1}{2\left(\frac{1}{2}\right)^2} - \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{2/4} - \frac{1}{2\left(\frac{1}{2}\right)}$$

$$= \frac{4}{2} - 1 = 2-1 = \boxed{1}$$

Symmetry and definite integrals:

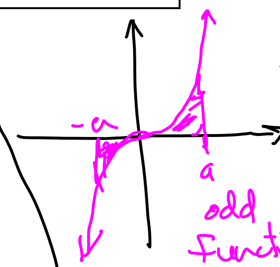
Suppose f is continuous on $[-a, a]$.

(a) If f is even (symmetric about ~~x-axis~~ y-axis, $f(-x) = f(x)$), then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.



even function

(b) If f is odd (symmetric about origin, $f(-x) = -f(x)$), then $\int_{-a}^{a} f(x)dx = 0$.



odd function

(net area = 0)

Example 14: $\int_{-3}^{3} x^2\, dx$

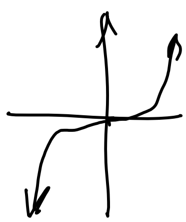
because $y = x^2$ is an even function (symmetric about the y-axis)

$$\int_{-3}^{3} x^2\, dx = 2\int_{0}^{3} x^2\, dx$$

$$= 2 \cdot \frac{x^3}{3}\Big|_0^3 = \frac{2(3)^3}{3} - 0 = \boxed{18}$$

Example 15: Find $\int_{-13}^{13} 7x^5 dx$

$$\int_{-13}^{13} 7x^5\, dx = \boxed{0}$$



odd function $y = 7x^5$