5.2: The Natural Logarithmic Function: Integration

Using the derivative of the natural logarithmic function to obtain an antiderivative:

Example 1: Find the derivative of $g(x) = \ln|x|$.

$$g(x) = \ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

($\ln|x|$ not defined for $x = 0$)

$$g'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

So, $\boxed{g'(x) = \frac{d}{dx}\ln|x| = \frac{1}{x}}$

Note that $f(x) = \ln x$ has the same derivative as $g(x) = \ln|x|$.

defined for $x > 0$ defined for $x \neq 0$

Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This means that $f(x) = \ln|x|$ is an antiderivative of $F(x) = \frac{1}{x}$.

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

Recall: The power rule for integrals $\int x^n dx = \frac{x^{n+1}}{n+1}$ had a restriction: ~~$n \neq 1$~~ $n \neq -1$. Now we can handle this case.

Ex: $$\int \frac{x^3 + 5x}{x^2}\,dx = \int \left(\frac{x^3}{x^2} + \frac{5x}{x^2}\right) dx$$

$$= \int \left(x + \frac{5}{x}\right) dx = \frac{x^2}{2} + 5\int \frac{1}{x}\,dx$$

$$= \boxed{\frac{x^2}{2} + 5\ln|x| + c}$$

Example 2: Determine $\int \frac{x^2}{x^3+4}\,dx$.

$u = x^3 + 4$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$\int \frac{x^2}{x^3+4}\,dx = \int \underbrace{x^2}\underbrace{(x^3+4)^{-1}}_{u^{-1}}\,\underbrace{dx}_{\frac{1}{3}du} = \frac{1}{3}\int u^{-1}\,du$$

$$= \frac{1}{3}\int \frac{1}{u}\,du = \boxed{\frac{1}{3}\ln|x^3+4| + C}$$

Check:

$$\frac{d}{dx}\left(\frac{1}{3}\ln(x^3+4)\right)$$

$$= \frac{1}{\cancel{3}} \cdot \frac{1}{x^3+4}(\cancel{3}x^2)$$

$$= \frac{x^2}{x^3+4} \checkmark$$

Example 3: Determine $\int \frac{7}{2-5x}\,dx$.

Example 4: Determine $\int_2^5 \frac{1}{3x}\,dx$.

Example 5: Determine $\int \frac{x^7 - x + 3x^4}{x^5}\,dx$.

Example 6: Find $\int \frac{(\ln x)^4}{x}\,dx$.

$$\int \underbrace{\frac{1}{x}\,dx}_{du}\underbrace{(\ln x)^4}_{u^4} = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{(\ln x)^5}{5} + C}$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

Check: $\frac{d}{dx}\left(\frac{1}{5}(\ln x)^5\right) = \frac{1}{5}(5)(\ln x)^4\left(\frac{1}{x}\right) = \frac{(\ln x)^4}{x}$

Example 7: Find $\int \frac{\ln(3x)}{x}\,dx$.

$$\int \underbrace{\frac{1}{x}\,dx}_{du}\underbrace{\ln(3x)}_{u} = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\ln(3x))^2}{2} + C}$$

$u = \ln(3x)$

$\frac{du}{dx} = \frac{1}{3x}(3) = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

Example 8: Find $\int \frac{x}{x^2-8}\,dx$.

$$\int \underbrace{x\,dx}_{\frac{1}{2}du}\underbrace{\left(\frac{1}{x^2-8}\right)}_{\frac{1}{u}} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|x^2-8| + C}$$

$u = x^2 - 8$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

Example 9: Find $\int \frac{4x^2-5x-12}{x^2-3}\,dx$.

Long division

$$\begin{array}{r} 4 \\ x^2+0x-3 \overline{)\,4x^2-5x-12} \\ -(4x^2-0x-12) \\ \hline -5x+0 \end{array}$$

$$\int \frac{4x^2-5x-12}{x^2-3}\,dx = \int\left(4 + \frac{-5x}{x^2-3}\right)dx$$

$$= \int 4\,dx - 5\int \frac{x}{x^2-3}\,dx$$

$$= 4x - 5\left(\frac{1}{2}\right)\int \frac{1}{u}\,du = 4x - \frac{5}{2}\ln|u| + C$$

$$= \boxed{4x - \frac{5}{2}\ln|x^2-3| + C}$$

$u = x^2 - 3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

Example 10: Find $\int \frac{4x^2-7x+1}{2x-3}\,dx$.

see archived notes for remaining examples

start by doing long division.

Integrating the remaining trigonometric functions:

Example 11: Determine $\int \tan x\,dx$.

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \underbrace{\left(\frac{1}{\cos x}\right)}_{\frac{1}{u}} \underbrace{\sin x\,dx}_{-du}$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C} = \boxed{\ln|\sec x| + C}$$

Check of Ex 11:

$$\frac{d}{dx}\left(-\ln(\cos x)\right) = -\frac{1}{\cos x}\left(\frac{d}{dx}(\cos x)\right)$$

$$= -\frac{1}{\cos x}(-\sin x) = \tan x \checkmark$$

Example 12: Determine $\int \cot x\,dx$.

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \boxed{\ln|\sin x| + C}$$

Example 13: Determine $\int \sec x\, dx$.

$$\int \sec x\, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec^2 x + \sec x \tan x)\, dx$

$$= \int \underbrace{\frac{1}{\sec x + \tan x}}_{\frac{1}{u}} \underbrace{(\sec^2 x + \sec x \tan x)\, dx}_{du}$$

$$= \int \frac{1}{u}\, du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

Check:

$\frac{d}{dx}\left(\ln(\sec x + \tan x)\right)$

$= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$

$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x\, \cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}} = \sec x$ ✓

Example 14: Determine $\int \csc x\, dx$.

$$\int \csc x\, dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx =$$

$u = \csc x + \cot x$

$\frac{du}{dx} = -\csc x \cot x - \csc^2 x$

$du = -(\csc x \cot x + \csc^2 x)\, dx$

$-du = (\csc x \cot x + \csc^2 x)\, dx$

$$= \int \underbrace{\frac{1}{\csc x + \cot x}}_{\frac{1}{u}} \underbrace{(\csc^2 x + \csc x \cot x)\, dx}_{-du}$$

$$= -\int \frac{1}{u}\, du = -\ln|u| + C = \boxed{-\ln|\csc x + \cot x| + C}$$

equivalent to $\ln|\csc x - \cot x| + C$

Homework probs from 5.2

#35) $\int (\cos(3\theta) - 1)\, d\theta$

$= \int \cos(3\theta)\, d\theta - \int 1\, d\theta$

$= \frac{1}{3}\int \cos(u)\, du - \theta + C$

$= \frac{1}{3}\sin(u) - \theta + C$

$= \frac{1}{3}\sin(3\theta) - \theta + C$

$u = 3\theta$

$\frac{du}{d\theta} = 3$

$du = 3\, d\theta$

$\frac{1}{3} du = d\theta$

Check:

$\frac{d}{d\theta}\left(\frac{1}{3}\sin 3\theta - \theta\right)$

$= \frac{1}{3}(\cos 3\theta)(3) - 1$ ✓