5.4: Exponential Functions: Differentiation and Integration

Short Review:

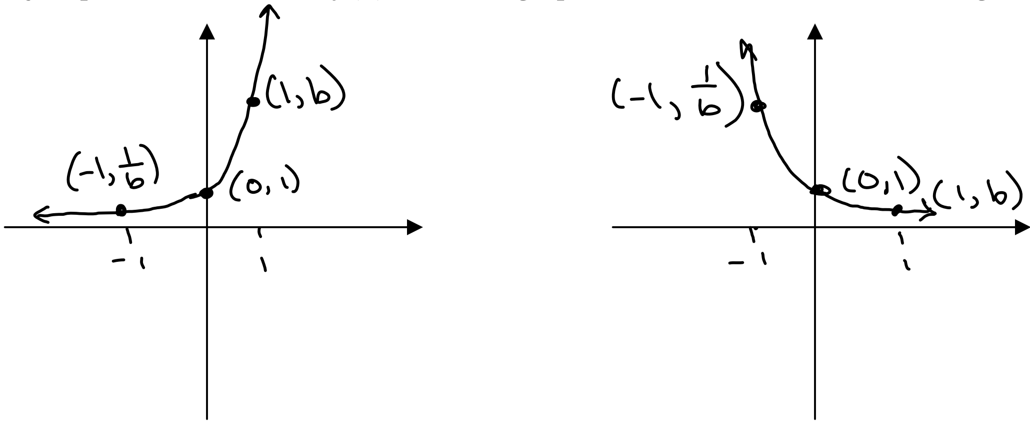
An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

why $b>0$? For $b<0$, $f(\frac{1}{2}) = b^{1/2}$ is not a real number.

For any exponential function $f(x) = b^x$, the graph looks like one of the following.

why $b \neq 1$?
$f(x) = 1^x = 1$
(constant function)

x	b^x
0	$b^0 = 1$
1	$b^1 = b$
-1	$b^{-1} = \frac{1}{b}$



$b > 1$ $\qquad\qquad$ $0 < b < 1$

Notice:

- Domain is $(-\infty, \infty)$.
- Range is $(0, \infty)$.
- Horizontal asymptote is $y = 0$.
- Always passes through the points $(0,1)$, $(1,b)$, $(-1, \frac{1}{b})$

The natural exponential function:

The number e can be defined in several ways.

> One definition of the number e:
>
> e is the number such that $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$\frac{0}{0}$ indeterminate form

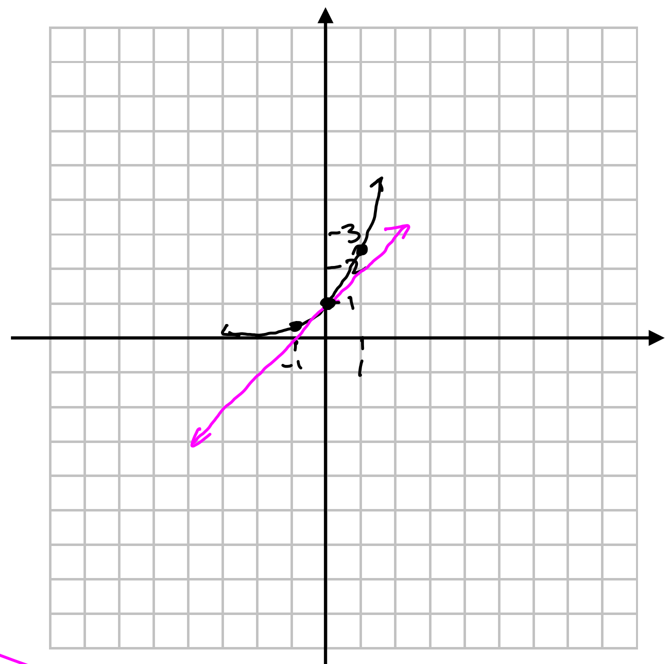
$$\lim_{h \to 0} \frac{e^h - e^0}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0}$$

$$\boxed{e \approx 2.718281828459}$$

The slope of the tangent line at the point $(0,1)$ is equal to 1.

The graph of $f(x)=e^x$**:**

x	$y=e^x$
0	$e^0=1$
1	$e^1=e\approx 2.7$
-1	$e^{-1}=\frac{1}{e}\approx\frac{1}{3}$



1^∞ indeterminate form

<u>Another definition of the number *e*:</u>

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \text{or, equivalently,} \quad e=\lim_{x\to 0}(1+x)^{1/x}$$

$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Let $h=\frac{1}{x}$.

as $x\to\infty$, $h\to 0$

Also $x=\frac{1}{h}$

so this limit becomes

$\lim_{h\to 0}(1+h)^{1/h}$

Derivatives of exponential functions:

$$\boxed{\frac{d}{dx}\left(e^x\right)=e^x}$$

<u>Example 1:</u> Find the derivative of $f(x)=-7e^x$.

$f'(x)=\boxed{-7e^x}$

Example 2: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

$f(x) = 5(e^x+7)^{1/2}$

$f'(x) = 5\left(\frac{1}{2}\right)(e^x+7)^{-1/2}\frac{d}{dx}(e^x+7) = \frac{5}{2}(e^x+7)^{-1/2}(e^x+0)$

$= \boxed{\dfrac{5e^x}{2\sqrt{e^x+7}}}$

Example 3: Find the derivative of $f(x) = e^x \sin x$.

Product Rule

$f'(x) = e^x\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(e^x)$

$= e^x\cos x + (\sin x)e^x = \boxed{e^x\cos x + e^x\sin x}$

$= \boxed{e^x(\cos x + \sin x)}$

Example 4: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

$g'(x) = e^{-7x}\frac{d}{dx}(-7x) + 6x^2 - 0 = e^{-7x}(-7) + 6x^2$

$= \boxed{-7e^{-7x} + 6x^2}$

Example 5: Find the derivative of $y = e^{x^2+4x}$.

$\frac{dy}{dx} = e^{x^2+4x}\frac{d}{dx}(x^2+4x) = \boxed{e^{x^2+4x}(2x+4)}$

Example 6: Find the derivative of $f(x) = \cos(e^x - x)$.

$f'(x) = [-\sin(e^x - x)]\frac{d}{dx}(e^x - x) = [-\sin(e^x-x)](e^x-1)$

$= \boxed{-(e^x-1)\sin(e^x-x)}$

Example 7: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0,9)$.

$f'(x) = 2(e^x+2)\frac{d}{dx}(e^x+2) = 2(e^x+2)(e^x) = 2e^x(e^x+2)$

Slope: $m = f'(0) = 2e^0(e^0+2) = 2(1)(1+2) = 2(3) = 6$

$y - y_1 = m(x - x_1)$

$y - 9 = 6(x - 0)$

$\boxed{y = 6x + 9}$ eqn of tangent line

Integration of exponential functions:

$$\boxed{\int e^x dx = e^x + c}$$

Example 8: Determine $\int (x^2 - 5e^x)\, dx$

$$\int (x^2 - 5e^x)\, dx = \int x^2 dx - 5\int e^x dx = \boxed{\frac{x^3}{3} - 5e^x + c}$$

Example 9: Find $\int e^{5t} dt$.

$$\int \underbrace{e^{5t}}_{e^u} \underbrace{dt}_{\frac{1}{5}du} = \frac{1}{5}\int e^u du$$

$$= \frac{1}{5}e^u + c$$

$$= \boxed{\frac{1}{5}e^{5t} + c}$$

$u = 5t$

$\frac{du}{dt} = 5$

$du = 5\,dt$

$\frac{1}{5}du = dt$

Check: $\frac{d}{dt}\left(\frac{1}{5}e^{5t}\right)$

$= \frac{1}{5}e^{5t}\frac{d}{dt}(5t) = \frac{1}{\cancel{5}}e^{5t}(\cancel{5})$

$= e^{5t}$

Example 10: Find $\int_1^3 e^{2x-3} dx$.

$$\int_1^3 e^{2x-3} dx = \frac{1}{2}\int_{-1}^{3} e^u du = \frac{1}{2}e^u\Big|_{-1}^{3}$$

$$= \boxed{\frac{1}{2}(e^3 - e^{-1})}$$

$u = 2x - 3$

$du = 2dx$

$\frac{1}{2}du = dx$

$x = 1 \Rightarrow u = 2(1) - 3 = -1$

$x = 3 \Rightarrow u = 2(3) - 3 = 3$

OR (without changing limits of integration)

$$\int_1^3 e^{2x-3} dx = \frac{1}{2}\int_{x=1}^{x=3} e^u du = \frac{1}{2}\cdot e^u\Big|_{x=1}^{x=3} = \frac{1}{2}e^{2x-3}\Big|_1^3$$

$$= \frac{1}{2}e^{2(3)-3} - \frac{1}{2}e^{2(1)-3} = \frac{1}{2}e^3 - \frac{1}{2}e^{-1} = \boxed{\frac{1}{2}(e^3 - e^{-1})}$$

Example 11: Find $\int t\overbrace{e^{t^2}}^{e^u}\underbrace{\, t \; dt}_{\frac{1}{2}du}$.

$u = t^2$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$\int t e^{t^2} dt = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \boxed{\frac{1}{2}e^{t^2} + c}$$

Example 12: Determine $\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx$.

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$$\int e^x \underbrace{(e^x+1)^{-1/3}}_{u^{-1/3}} dx \quad (e^x\,dx = du) \quad = \int u^{-1/3} du = \frac{u^{2/3}}{2/3} + c$$

$$= \frac{3}{2}u^{2/3} + c$$

$$= \boxed{\frac{3}{2}(e^x+1)^{2/3} + c} = \frac{3}{2}\sqrt[3]{(e^x+1)^2} + c$$

Example 13: Determine $\int \frac{e^x - e^{-x}}{e^{3x}}\,dx$

$$\int \frac{e^x - e^{-x}}{e^{3x}}\,dx = \int\left(\frac{e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}}\right)dx = \int\left(e^{-2x} - e^{-4x}\right)dx$$

$$= \int e^{-2x}dx - \int e^{-4x}dx$$

2nd integral

$u = -4x$

$du = -4dx$

$-\frac{1}{4}du = dx$

1st integral

$u = -2x$

$du = -2dx$

$-\frac{1}{2}du = dx$

$$= -\frac{1}{2}\int e^u du - \left(-\frac{1}{4}\right)\int e^u du$$

$$= \boxed{-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x} + c}$$