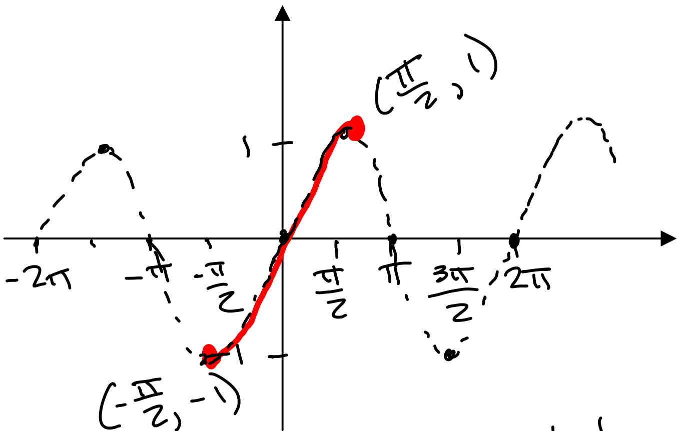
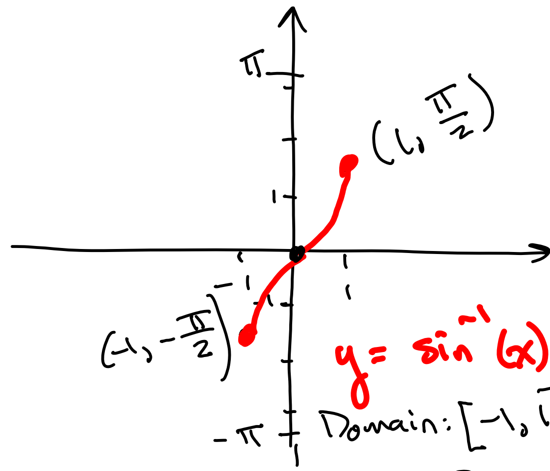
5.6: Inverse Trigonometric Functions – Differentiation

Because none of the trigonometric functions are one-to-one, none of them have an inverse function. To overcome this problem, the domain of each function is restricted so as to produce a one-to-one function.

Inverse sine function:

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \ \text{ and } \ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$





Domain: $[-1, 1]$

Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \sin x$ Fails the horizontal line test - it is not one-to-one (does not have an inverse function)

Properties of the inverse sine function:

$$\sin^{-1}(\sin x) = x \ \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \ \text{ for } -1 \le x \le 1$$

Recall: If f and g are inverses of one another, then $f(g(x)) = x$ and $g(f(x)) = x$ for all x (in domain)

in other words, $f^{-1}(f(x)) = f(f^{-1}(x)) = x$

Example 1: Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{2}\right)$.

(a) Find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$:

want θ so that $\sin\theta = \frac{\sqrt{3}}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\theta = \frac{\pi}{3}$ works.

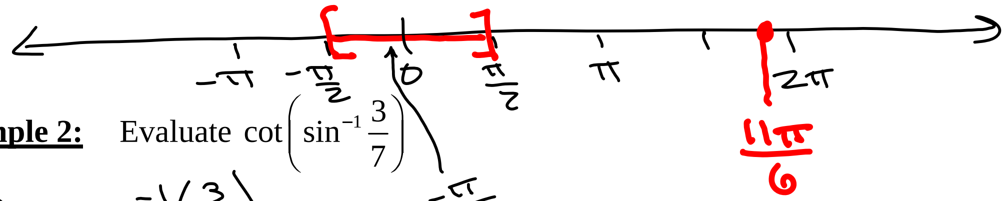
So $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

(b) want θ so that $\sin\theta = -\frac{1}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\theta = -\frac{\pi}{6}$ works: so $\boxed{\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}}$

see next page

What about $\theta = \frac{11\pi}{6}$? It's not in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

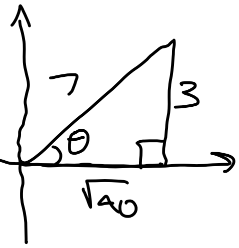


Example 2: Evaluate $\cot\left(\sin^{-1}\frac{3}{7}\right)$

Let $\theta = \sin^{-1}\left(\frac{3}{7}\right)$.

we know $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

and $\sin\theta = \frac{3}{7}$.



$3^2 + a^2 = 7^2$

$9 + a^2 = 49$

$a^2 = 40$

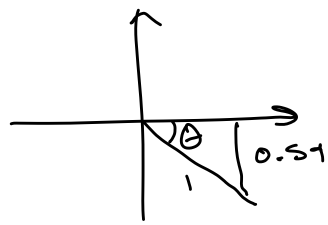
$a = \sqrt{40}$

$\cot\left(\sin^{-1}\left(\frac{3}{7}\right)\right) = \cot\theta = \boxed{\frac{\sqrt{40}}{3}}$

Example 3: Evaluate $\sin\left(\sin^{-1}(-0.54)\right)$.

$\theta = \sin^{-1}(-0.54)$

$\sin\theta = -0.54$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\sin(\sin^{-1}(-0.54)) = \sin\theta = \boxed{-0.54}$

Example 4: Evaluate $\sin\left(\sin^{-1} 2\right)$.

Let $\theta = \sin^{-1} 2$. Then $\sin\theta = 2$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

impossible!

So, $\boxed{\sin(\sin^{-1} 2) \text{ is not defined (does not exist)}}$

Example 5: Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

$\sin^{-1}(\sin\frac{\pi}{4})$

$= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\boxed{\frac{\pi}{4}}$

Ex $5\frac{1}{2}$: $\sin\left(\sin^{-1}\left(\frac{\pi}{4}\right)\right)$

$\boxed{\frac{\pi}{4}}$

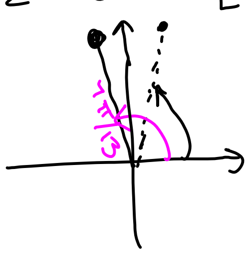
Example 6: Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

(Find θ so ~~that~~ that $\sin\theta = \frac{1}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$)

Ex $6\frac{3}{4}$: $\sin^{-1}\left(\sin\frac{7\pi}{13}\right)$

$= \pi - \frac{7\pi}{13} = \boxed{\frac{6\pi}{13}}$



Ex $6\frac{1}{2}$: $\sin\left(\sin^{-1}\left(\frac{5\pi}{6}\right)\right)$

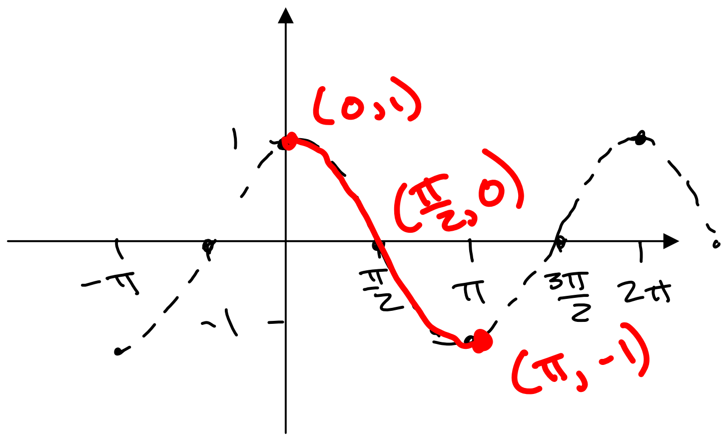
$\boxed{\text{does not exist}}$

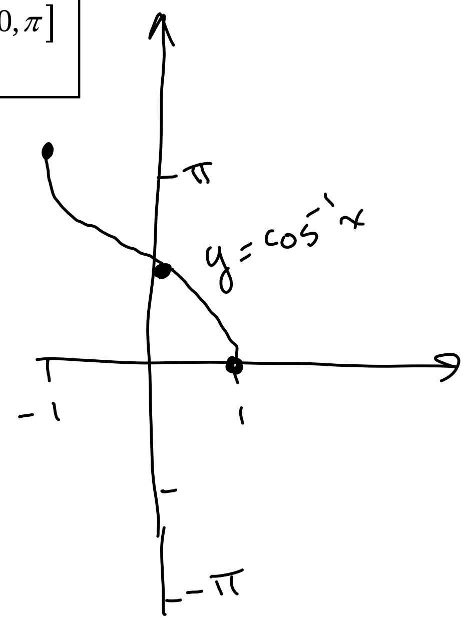
$\theta = \sin^{-1}\left(\frac{5\pi}{6}\right)$

$\sin\theta = \frac{5\pi}{6}$ but $\frac{5\pi}{6} > 1$

Inverse cosine function:

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \ \text{ and } \ y \in [0, \pi]$$





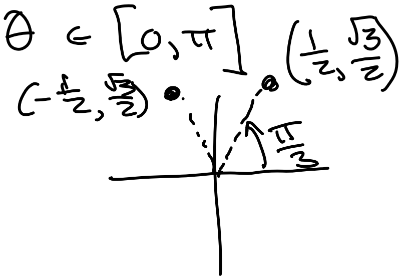
Properties of the inverse cosine function:

$$\cos^{-1}(\cos x) = x \ \text{ for } \ 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \ \text{ for } \ -1 \le x \le 1$$

Example 7: Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$. Then $\cos\theta = -\frac{1}{2}$ and $\theta \in [0, \pi]$

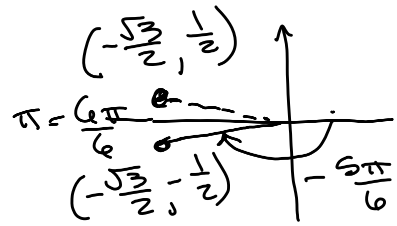
$\theta = \frac{2\pi}{3}$ works.



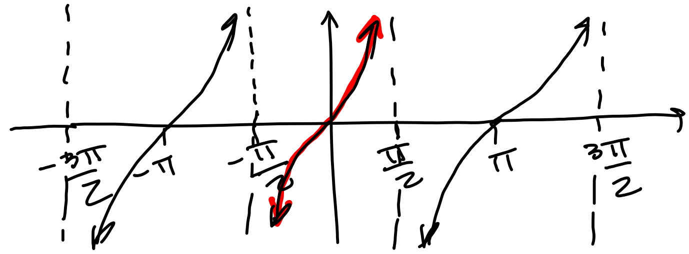
$$\boxed{\text{So } \cos^{-1}\left(-\tfrac{1}{2}\right) = \frac{2\pi}{3}}$$

Example 8: Evaluate $\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right)$

$$\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \boxed{\frac{5\pi}{6}}$$



$y = \tan x$



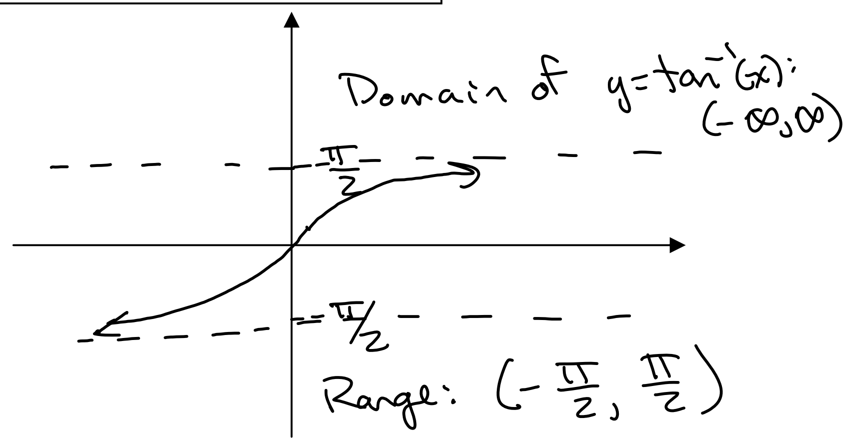
Inverse tangent function:

$y = \tan^{-1} x$ if and only if $\tan y = x$ and $y \in \cancel{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$ $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\lim\limits_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim\limits_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$

Graph of $y = \tan^{-1} x$:

Know this graph!!



Inverse secant, cosecant, and cotangent functions:

These are not used as often, and are not defined consistently. Our book defines them as follows:

$y = \cot^{-1} x$ if and only if $\cot y = x$ and $y \in (0, \pi)$

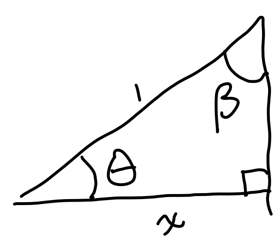
$y = \csc^{-1} x$ if and only if $\csc y = x$ and $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$y = \sec^{-1} x$ if and only if $\sec y = x$ and $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Let

$\theta = \cos^{-1} x$.

Then $\cos\theta = x = \frac{x}{1}$



$\theta + \beta = \frac{\pi}{2}$

$\beta = \sin^{-1} x$

Then $\sin\beta = x = \frac{x}{1}$

An important identity:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

(can show it works for angles not in Quad I also.)

Differentiation of the inverse sine function:

Since $y = \sin x$ is continuous and differentiable, so is $y = \sin^{-1} x$.

We want to find its derivative.

$y = \sin^{-1} x$. Find $\frac{dy}{dx}$

$\sin y = x$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Implicit diff.:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$(\cos y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

we need to write $\cos y$ in terms of x:

we know that $\cos^2 y + \sin^2 y = 1$ (Pythagorean Identity)

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm\sqrt{1-\sin^2 y}.$$

We know $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Quadrants I, IV). So $\cos y$ must be positive

So $\cos y = +\sqrt{1-\sin^2 y}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

substituting $\sin y = x$

★ memorize this

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

Example 9: Differentiate $f(x) = \sin^{-1}(2x-7)$.

$$f'(x) = \frac{1}{\sqrt{1-(2x-7)^2}} \frac{d}{dx}(2x-7)$$

(chain Rule)

$$= \frac{1}{\sqrt{1-(4x^2-28x+49)}}(2)$$

$$= \boxed{\frac{2}{\sqrt{-4x^2+28x-48}}} \quad \text{OR} \quad \boxed{\frac{2}{\sqrt{1-(2x-7)^2}}}$$

Differentiation of the inverse cosine function:

Know it!

$$\boxed{\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

Important:
Equivalent names/notation

$\tan^{-1} x = \arctan x$

$\cos^{-1} x = \arccos x$

$\sin^{-1} x = \arcsin x$

$e^x = \exp(x)$

Differentiation of the inverse ~~cosine~~ tangent function:

Know it

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

Derivatives of other inverse trigonometric functions:

memorize these 3

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

no need to memorize for $\csc^{-1}$, $\sec^{-1}$, $\cot^{-1}$

Example 10: Find the derivative of $f(x) = e^{\tan^{-1} 2x}$.

$$f'(x) = e^{\tan^{-1}(2x)} \frac{d}{dx}\left(\tan^{-1}(2x)\right) \quad \text{(chain rule)}$$

$$= e^{\tan^{-1}(2x)}\left(\frac{1}{1+(2x)^2}\frac{d}{dx}(2x)\right) \quad \text{(chain rule)}$$

$$= e^{\tan^{-1}(2x)}\left(\frac{1}{1+4x^2}\right)(2) = \boxed{\frac{2e^{\tan^{-1}(2x)}}{1+4x^2}}$$

Example 11: Find the derivative of $y = \csc^{-1}(\tan x)$.

$$\frac{dy}{dx} = -\frac{1}{|\tan x|\sqrt{\tan^2 x - 1}}\frac{d}{dx}(\tan x)$$

using $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$

$$= -\frac{1}{|\tan x|\sqrt{\tan^2 x - 1}} \cdot \sec^2 x = \boxed{-\frac{\sec^2 x}{|\tan x|\sqrt{\tan^2 x - 1}}}$$

Example 12: Find the derivative of $f(x) = x^3 \arccos 2x$.

$$f'(x) = x^3 \frac{d}{dx}\left(\cos^{-1}(2x)\right) + \left(\cos^{-1}(2x)\right)\frac{d}{dx}\left(x^3\right) \quad \text{[Product Rule]}$$

$$= x^3\left(-\frac{1}{\sqrt{1-(2x)^2}}\right)\underbrace{\frac{d}{dx}(2x)}_{2} + \left(\cos^{-1}(2x)\right)\left(3x^2\right) = \boxed{-\frac{2x^3}{\sqrt{1-4x^2}} + 3x^2\cos^{-1}(2x)}$$

Example 13: Find the equation of the line tangent to the graph of $f(x) = \arctan x$ at the point where $x = -1$.

$$f(x) = \arctan x = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$m = f'(-1) = \frac{1}{1+(-1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

Find y-value: $f(-1) = \arctan(-1) = -\frac{\pi}{4}$

$$y - y_1 = m(x - x_1)$$

$$y - \left(-\frac{\pi}{4}\right) = \frac{1}{2}(x - (-1))$$

$$\Rightarrow y + \frac{\pi}{4} = \frac{1}{2}(x+1)$$

$$y = \frac{1}{2}x + \frac{1}{2} - \frac{\pi}{4}$$

$$\boxed{y = \frac{1}{2}x + \frac{2-\pi}{4}}$$