5.7: Inverse Trigonometric Functions – Integration

Two important integration rules come from the inverse trigonometric differentiation rules.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

Recall: $\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

Example 1: Find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$.

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \underbrace{\frac{1}{\sqrt{1-(3x)^2}}}_{\frac{1}{\sqrt{1-u^2}}}\,\underbrace{dx}_{\frac{1}{3}du}$$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}(u) + C = \boxed{\frac{1}{3}\sin^{-1}(3x) + C}$$

Check it: $\frac{d}{dx}\left(\frac{1}{3}\sin^{-1}(3x)\right) =$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{1-(3x)^2}}\,\frac{d}{dx}(3x)$$

$$= \frac{1}{\cancel{3}}\cdot\frac{1}{\sqrt{1-9x^2}}(\cancel{3}) \checkmark$$

Example 2: Find $\int \frac{1}{x^2+7}\,dx$.

$$\int \frac{1}{x^2+7}\,dx = \int \frac{1}{7\left(\frac{x^2}{7}+1\right)}\,dx$$

$$= \frac{1}{7}\int \frac{1}{\frac{x^2}{7}+1}\,dx = \frac{1}{7}\int \underbrace{\frac{1}{\left(\frac{x}{\sqrt{7}}\right)^2+1}}_{\frac{1}{u^2+1}}\,\underbrace{dx}_{\sqrt{7}\,du}$$

$u^2 = \frac{x^2}{7}$

$u = \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}x$

$\frac{du}{dx} = \frac{1}{\sqrt{7}}$

$du = \frac{1}{\sqrt{7}}dx$

$\sqrt{7}\,du = dx$

$$= \frac{1}{7}\cdot\sqrt{7}\int \frac{1}{u^2+1}\,du$$

$$= \frac{\sqrt{7}}{7}\tan^{-1}(u) + C = \boxed{\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$$

$$\boxed{= \frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{7}x}{7}\right) + C} = \boxed{\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$$

More general forms of these integration rules are

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\,.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

Example 3: Find $\int \frac{4x}{x^4+25}\,dx$.

$$\int \frac{4x}{(x^2)^2+25}\,dx = 4\int \frac{x}{(x^2)^2+25}\,dx$$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= 4\cdot\frac{1}{2}\int \frac{1}{u^2+25}\,du$$

$$= 2\int \frac{1}{u^2+5^2}\,du = 2\left(\frac{1}{5}\right)\tan^{-1}\left(\frac{u}{5}\right)+C$$

using $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$$= \boxed{\frac{2}{5}\tan^{-1}\left(\frac{x^2}{5}\right)+C}$$

Check: $\frac{d}{dx}\left(\frac{2}{5}\tan^{-1}\left(\frac{1}{5}x^2\right)\right)$

$$= \frac{2}{5}\cdot\frac{1}{1+\left(\frac{1}{5}x^2\right)^2}\frac{d}{dx}\left(\frac{1}{5}x^2\right)$$

$$= \frac{2}{5}\cdot\frac{1}{1+\frac{x^4}{25}}\cdot\frac{1}{5}(2x)$$

$$= \frac{4x}{25+x^4} \checkmark$$

Another antiderivative:

$$\boxed{\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x|+C}$$

Example 4: Find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

$$\int \frac{1}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}}\,dx = \int \frac{1}{x\sqrt{4}\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx$$

$$= \frac{1}{\sqrt{4}}\int \frac{1}{\underbrace{x}_{\frac{1}{x}}\underbrace{\sqrt{\left(\frac{x}{2}\right)^2-1}}_{\frac{1}{\sqrt{u^2-1}}}}\underbrace{dx}_{2du}$$

$u = \frac{x}{2} = \frac{1}{2}x$

$du = \frac{1}{2}dx$

$2\,du = dx$

$x = 2u$

$$= \frac{1}{2}\cdot 2\int \frac{1}{x\sqrt{u^2-1}}\,du$$

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$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \int \frac{1}{2u\sqrt{u^2-1}}\,du = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du$$

$$= \frac{1}{2}\sec^{-1}|u| + C$$

$$= \boxed{\frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right) + C}$$

Example 5: Find $\int \frac{1}{4x^2-12x+17}\,dx$.

Complete the Square:

$$4x^2-12x+17$$

$$= (4x^2-12x)+17$$

$$= 4(x^2-3x)+17$$

$$= 4\left(x^2-3x+\frac{9}{4}\right)+17-4\left(\frac{9}{4}\right)$$

$$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

$$= 4\left(x-\frac{3}{2}\right)^2+17-9$$

$$= 4\left(x-\frac{3}{2}\right)^2+8$$

$$\int \frac{1}{4\left(x-\frac{3}{2}\right)^2+8}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x-\frac{3}{2}\right)^2+2}\,dx$$

$u = x-\frac{3}{2}$

$du = 1\,dx$

$$= \frac{1}{4}\int \frac{1}{u^2+(\sqrt{2})^2}\,du$$

$$= \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C = \boxed{\frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x-3/2}{\sqrt{2}}\right)+C}$$

Example 6: Find $\int \frac{x-7}{\sqrt{5-4x^2}}\,dx$.

$$\int \frac{x-7}{\sqrt{5-4x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{5-4x^2}}\,dx - \int \frac{7}{\sqrt{5-4x^2}}\,dx$$

$u = 5-4x^2$

$\frac{du}{dx} = -8x$

$du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$$= \int \underbrace{x\,\underbrace{(5-4x^2)^{-1/2}}_{u^{-1/2}}\,dx}_{-\frac{1}{8}du} - 7\int \frac{1}{\sqrt{5-(2x)^2}}\,dx$$

$u = 2x$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= -\frac{1}{8}\int u^{-1/2}\,du - 7\left(\frac{1}{2}\right)\int \frac{1}{\sqrt{5-u^2}}\,du$$

Use $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$ with $a=\sqrt{5}$

$$= -\frac{1}{8}\frac{u^{1/2}}{1/2} - \frac{7}{2}\sin^{-1}\left(\frac{u}{\sqrt{5}}\right)+C$$

$$= -\frac{2}{8}u^{1/2} - \frac{7}{2}\sin^{-1}\left(\frac{u}{\sqrt{5}}\right)+C$$

$$= \boxed{-\frac{1}{4}\sqrt{5-4x^2} - \frac{7}{2}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right)+C}$$