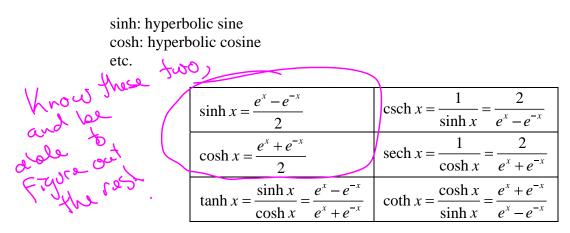
5.8: Hyperbolic Functions

The unit circle $x^2 + y^2 = 1$ is used in trigonometry to define the trigonometric functions. Often they are called "circular functions." In a similar manner, the unit hyperbola $x^2 - y^2 = 1$ can be used to define functions that have many similar characteristics as the circular trigonometric functions. Such functions are called "hyperbolic functions." These hyperbolic functions can also be defined in terms of the natural exponential function.



Recall: odd and even functions

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We can also think of decomposing e^x into odd and even parts:

Example 1: Evaluate (a) cosh 2 (b) tanh 0 (c) sinh 1

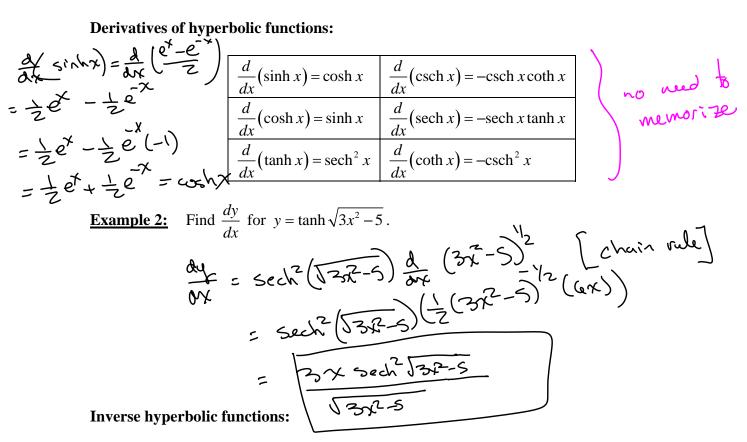
(a)
$$\cosh 2 = \frac{e^2 + e^{-2}}{2}$$
 atc

Hyperbolic functions have many uses in science and engineering applications.

Hyperbolic identities:

Several identities involving hyperbolic functions are similar to the usual trigonometric identities, but there are a few important differences.

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$	
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$	
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$		
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$		
$\sinh 2t = 2\sinh t \cosh t$		
$\cosh 2t = \cosh^2 t + \sinh^2 t$		



The functions sinh and tanh are one-to-one functions, which means they have inverse functions. The function cosh, however, is not one-to-one. But, restricting the domain of cosh to $[0,\infty)$ defines a one-to-one function.

> $y = \sinh^{-1} x$ if and only if $\sinh y = x$ $y = \cosh^{-1} x$ if and only if $\cosh y = x$ and $y \ge 0$ $y = \tanh^{-1} x$ if and only if $\tanh y = x$

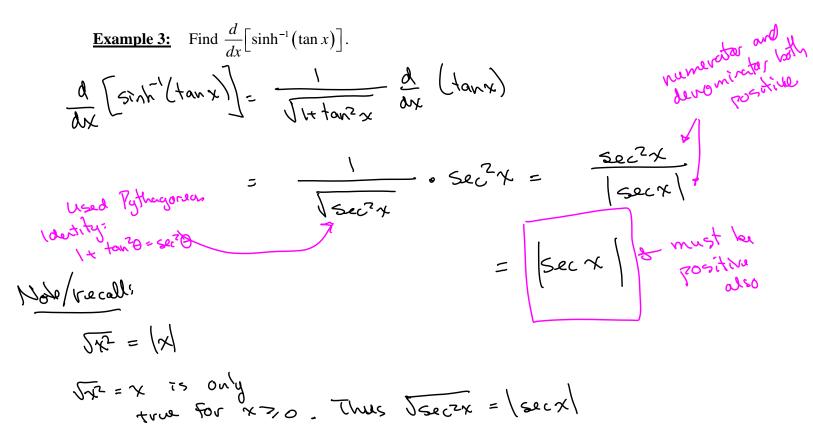
Because the hyperbolic functions are defined in terms of the natural exponential function, it should come as no surprise that the inverse hyperbolic functions are defined in terms of the natural logarithmic function.

$$\begin{array}{c} (1) \quad \text{Derivation `d} \\ \text{Sint' } x \quad \text{is on} \\ \text{Sint' } x \quad \text{is on} \\ \text{last page of} \\ \text{votes} \end{array} \end{array} \qquad \begin{array}{c} \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), x \in \mathbb{R} \\ (1) \quad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1 \\ \text{tanh}^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right), -1 < x < 1 \end{array} \qquad \begin{array}{c} \text{No} \quad need \\ \text{to} \\ \text{Memorize} \end{array}$$

Derivatives of Inverse Hyperbolic Functions

no need	$\frac{d}{dx}\left(\sinh^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{ x \sqrt{x^2+1}}$
to memoria)	$\frac{d}{dx}\left(\cosh^{-1}x\right) = \frac{1}{\sqrt{x^2 - 1}}$	$\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right) = -\frac{1}{x\sqrt{1-x^2}}$
/	$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$	$\frac{d}{dx}\left(\coth^{-1}x\right) = \frac{1}{1-x^2}$

<u>Note</u>: Even though the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ look identical, they are really different because these inverse functions are defined on different domains: |x| < 1 for $\tanh^{-1} x$, and |x| > 1 for $\coth^{-1} x$.



Example 4:
$$\int \frac{3}{\sqrt{x^2 - 1}} dx$$

$$\int \frac{3}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{x^2 - 1}} dx$$

Example 5: Evaluate
$$\int \frac{4dx}{\sqrt{9x^2 - 1}}$$
.

$$\int \frac{4}{\sqrt{9x^2 - 1}} = 4 \int \frac{1}{\sqrt{(3x)^2 - 1}} dx$$

$$= 4 \left(\frac{1}{3}\right) \int \frac{1}{\sqrt{12^2 - 1}} du$$

$$= \frac{4}{3} \cosh^{-1}(u) + c = \left[\frac{4}{3} \cosh^{-1}(3x) + c\right]$$

Example 6: Evaluate $\int \coth x \, dx$

$$\int \cot x \, dx = \int \frac{\cosh x}{\sinh x} \, dx$$

$$= \int \frac{1}{\sin hx} \cdot \cosh x \, dx$$

$$= \int \frac{1}{\sin hx} \cdot \cosh x \, dx$$

$$= \int \frac{1}{\sin hx} \cdot \cosh x \, dx$$

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How to dorive the formula sinh
$$x = ln(x+ix_{i1}), x \in \mathbb{R}$$
:
From definition of sinhx,
 $y = \sinh x = \frac{e^{x} - e^{2}}{2}$
switch the variables; $x = \frac{e^{x} - e^{2}}{2}$
to get the inverse function, use solve for y:
 $2x = e^{2} - e^{2}$
 $Multiply both sides (ay et!:
 $2xe^{2} = (e^{x})^{2} - e^{2}bt$
 $0 = (e^{x})^{2} - 2xe^{2} - 1$
This equation is guodratic in e^{2} :
(It you let $u = e^{2}$, you get $u^{2} - exist - 1$
So apply guodratic formula, using $a = 1, b = -2x, c - 1$
 $selections di
 $u^{2} = \frac{-bt}{2} \frac{1}{2} \frac{1}{$$$

So, we now have
$$e^{t} = \chi + \sqrt{\chi^{2} + 1}$$

Apply $\ln t_{0}$ both sides:
 $l_{n}(e^{t}) = l_{n}(\chi + \sqrt{\chi^{2} + 1})$
 $y = l_{n}(\chi + \sqrt{\chi^{2} + 1})$
Therefore, $\sinh^{-1}(\chi) = l_{n}(\chi + \sqrt{\chi^{2} + 1})$.

 $\overline{\mathbf{A}}$