5.8: Hyperbolic Functions

The unit circle $x^2 + y^2 = 1$ is used in trigonometry to define the trigonometric functions. Often they are called "circular functions." In a similar manner, the unit hyperbola $x^2 - y^2 = 1$ can be used to define functions that have many similar characteristics as the circular trigonometric functions. Such functions are called "hyperbolic functions." These hyperbolic functions can also be defined in terms of the natural exponential function.

sinh: hyperbolic sine
cosh: hyperbolic cosine
etc.

Know these two, and be able to figure out the rest.

$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

<u>Recall</u>: odd and even functions

We can also think of decomposing e^x into odd and even parts:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x$$

The odd part is $\dfrac{e^x - e^{-x}}{2} = \sinh x$ and the even part is $\dfrac{e^x + e^{-x}}{2} = \cosh x$.

Example 1: Evaluate (a) $\cosh 2$ (b) $\tanh 0$ (c) $\sinh 1$

(a) $\cosh 2 = \dfrac{e^2 + e^{-2}}{2}$

etc

Hyperbolic functions have many uses in science and engineering applications.

Hyperbolic identities:

Several identities involving hyperbolic functions are similar to the usual trigonometric identities, but there are a few important differences.

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh 2t = 2 \sinh t \cosh t$	
$\cosh 2t = \cosh^2 t + \sinh^2 t$	

Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$
$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}(-1)$$
$$= \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \cosh x$$

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

no need to memorize

Example 2: Find $\frac{dy}{dx}$ for $y = \tanh\sqrt{3x^2 - 5}$.

$$\frac{dy}{dx} = \operatorname{sech}^2\left(\sqrt{3x^2-5}\right)\frac{d}{dx}(3x^2-5)^{1/2}$$

[chain rule]

$$= \operatorname{sech}^2\left(\sqrt{3x^2-5}\right)\left(\frac{1}{2}(3x^2-5)^{-1/2}(6x)\right)$$

$$= \boxed{\frac{3x \operatorname{sech}^2\sqrt{3x^2-5}}{\sqrt{3x^2-5}}}$$

Inverse hyperbolic functions:

The functions $\sinh$ and $\tanh$ are one-to-one functions, which means they have inverse functions. The function $\cosh$, however, is not one-to-one. But, restricting the domain of $\cosh$ to $[0, \infty)$ defines a one-to-one function.

$$y = \sinh^{-1} x \text{ if and only if } \sinh y = x$$

$$y = \cosh^{-1} x \text{ if and only if } \cosh y = x \text{ and } y \geq 0$$

$$y = \tanh^{-1} x \text{ if and only if } \tanh y = x$$

Because the hyperbolic functions are defined in terms of the natural exponential function, it should come as no surprise that the inverse hyperbolic functions are defined in terms of the natural logarithmic function.

(Derivation of $\sinh^{-1}x$ is on last page of notes)

$\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right),\ x\in\mathbb{R}$
$\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right),\ x\geq 1$
$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1<x<1$

no need to memorize

Derivatives of Inverse Hyperbolic Functions

no need to memorize

$\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right)=-\frac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}$	$\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{1-x^2}$

Note: Even though the derivativse of $\tanh^{-1}x$ and $\coth^{-1}x$ look identical, they are really different because these inverse functions are defined on different domains: $|x|<1$ for $\tanh^{-1}x$, and $|x|>1$ for $\coth^{-1}x$.

Example 3: Find $\frac{d}{dx}\left[\sinh^{-1}(\tan x)\right]$.

$$\frac{d}{dx}\left[\sinh^{-1}(\tan x)\right]=\frac{1}{\sqrt{1+\tan^2 x}}\frac{d}{dx}(\tan x)$$

numerator and denominator both positive

$$=\frac{1}{\sqrt{\sec^2 x}}\cdot\sec^2 x=\frac{\sec^2 x}{|\sec x|}$$

Used Pythagorean Identity: $1+\tan^2\theta=\sec^2\theta$

$$=\boxed{|\sec x|}$$

$\therefore$ must be positive also

Note/recall:

$\sqrt{x^2}=|x|$

$\sqrt{x^2}=x$ is only true for $x\geq 0$. Thus $\sqrt{\sec^2 x}=|\sec x|$

Example 4: $\int \frac{3}{\sqrt{x^2-1}}\,dx$

$$\int \frac{3}{\sqrt{x^2-1}}\,dx = 3\int \frac{1}{\sqrt{x^2-1}}\,dx = \boxed{3\cosh^{-1}x + C}$$

Example 5: Evaluate $\int \frac{4dx}{\sqrt{9x^2-1}}$.

$$u = 3x \qquad \frac{du}{dx} = 3 \qquad du = 3\,dx \qquad \frac{1}{3}du = dx$$

$$\int \frac{4\,dx}{\sqrt{9x^2-1}} = 4\int \frac{1}{\sqrt{(3x)^2-1}}\,dx$$

$$= 4\left(\frac{1}{3}\right)\int \frac{1}{\sqrt{u^2-1}}\,du$$

$$= \frac{4}{3}\cosh^{-1}(u) + C = \boxed{\frac{4}{3}\cosh^{-1}(3x) + C}$$

Example 6: Evaluate $\int \coth x\,dx$

$$u = \sinh x \qquad \frac{du}{dx} = \cosh x \qquad du = \cosh x\,dx$$

$$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$$

$$= \int \underbrace{\frac{1}{\sinh x}}_{\frac{1}{u}} \cdot \underbrace{\cosh x\,dx}_{du}$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sinh x| + C}$$

How to derive the formula $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), x \in \mathbb{R}$:

From definition of $\sinh x$,

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

switch the variables: $$x = \frac{e^y - e^{-y}}{2}$$

To get the inverse function, we solve for y:

$$2x = e^y - e^{-y}$$

Multiply both sides by e^y:

$$2xe^y = (e^y)^2 - \underbrace{e^{-y}e^y}_{1}$$

$$0 = (e^y)^2 - 2xe^y - 1$$

This equation is quadratic in e^y:
(if you let $u = e^y$, you get $u^2 - (2x)u - 1$)

So apply quadratic formula, using $a = 1$, $b = -2x$, $c = -1$

Solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm \sqrt{4}\sqrt{x^2+1}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= \frac{2x}{2} \pm \frac{2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

We throw out $x - \sqrt{x^2+1}$ because e^y can't be negative.
(If $x \le 0$, then both terms of $x - \sqrt{x^2+1}$ are negative;
If $x > 0$, then $\sqrt{x^2+1} > \sqrt{x^2} = x$, so 2nd term of $x - \sqrt{x^2+1}$ is bigger, making the whole thing negative).

See next page

So, we now have $e^y = x + \sqrt{x^2+1}$

Apply ln to both sides:

$$\ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

Therefore, $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

$\boxtimes$