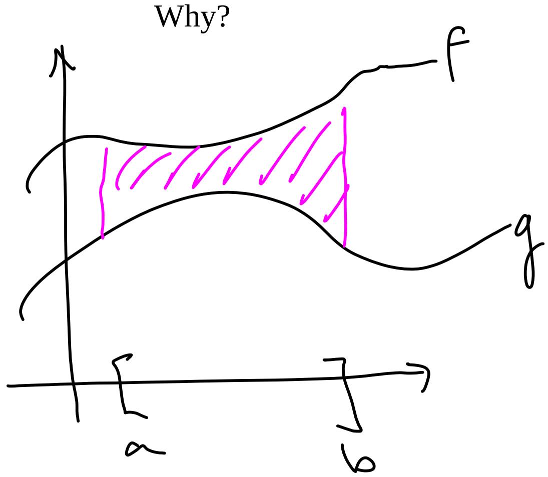
7.1: Area of a Region Between Two Curves

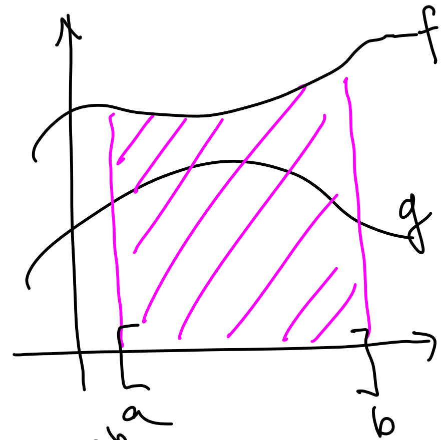
Because the definite integral represents the "net" area under a curve, we can use integration to find the area between curves.

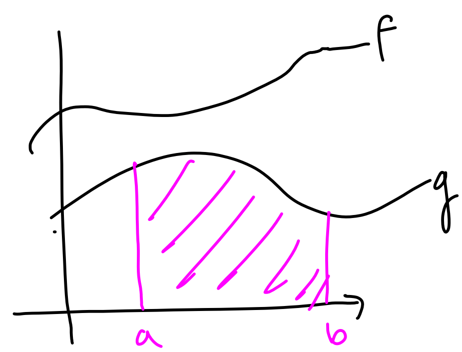
If f and g are continuous and $f(x) \geq g(x)$ on $[a,b]$, then the area between $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by

$$\text{Area } = \int_a^b [f(x) - g(x)]dx$$

Why?





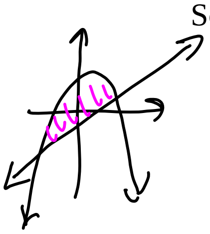


Area between curves $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

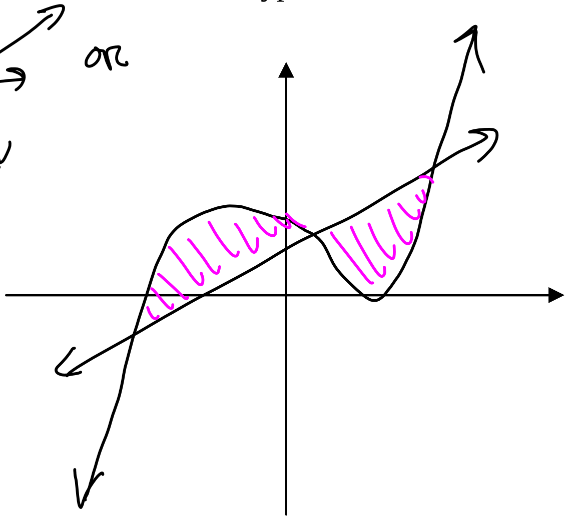
$= \int_a^b [f(x) - g(x)]\,dx$

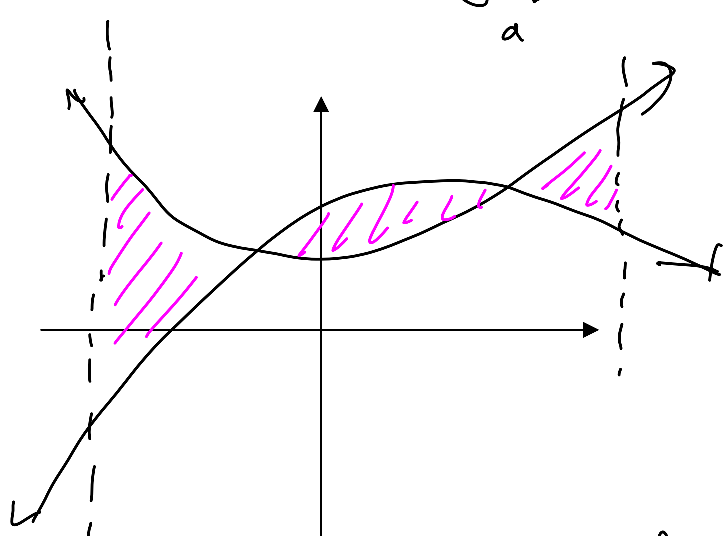
Note: If $g(x) \geq f(x)$ on $[a,b]$, then $\int_a^b [f(x) - g(x)]dx$ is negative.

Some different types of area scenarios:



or



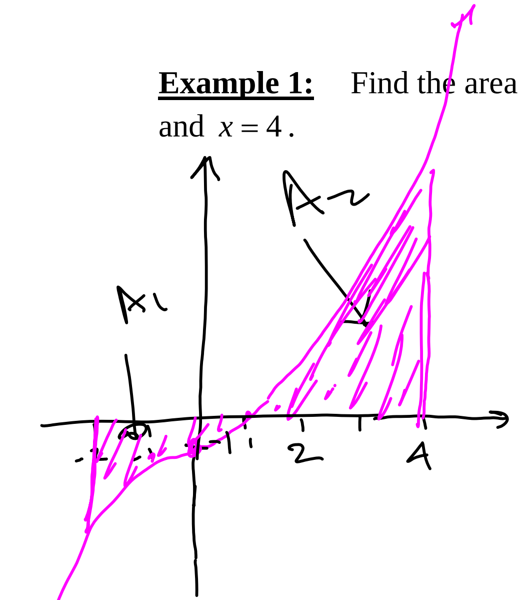


Find the area enclosed by 2 curves

Area is bounded by 2 or more curves and by 2 vertical lines

$\nearrow g(x)=0$

Example 1: Find the area of the region bounded by $f(x)=x^3-1$ and the lines $y=0$, $x=-2$, and $x=4$.



Area = $A_1 + A_2$

$$= \int_{-2}^{1}\left[0-(x^3-1)\right]dx + \int_{1}^{4}\left[(x^3-1)-0\right]dx$$

$$= -\int_{-2}^{1}(x^3-1)dx + \int_{1}^{4}(x^3-1)dx$$

$$= -\left[\frac{x^4}{4}-x\right]\Bigg|_{-2}^{1} + \left[\frac{x^4}{4}-x\right]\Bigg|_{1}^{4}$$

$$= -\left[\frac{1^4}{4}-1-\left(\frac{(-2)^4}{4}-(-2)\right)\right]+\left[\frac{4^4}{4}-4-\left(\frac{1^4}{4}-1\right)\right]$$

$$= -\left[\frac{1}{4}-1-(4+2)\right]+\left[64-4-\frac{1}{4}+1\right]$$

$$= -\left[\frac{1}{4}-1-6\right]+\left[60+\frac{3}{4}\right] = -\left[-\frac{27}{4}\right]+\left[\frac{243}{4}\right]$$

$$= \frac{27}{4}+\frac{243}{4}$$

$$= \frac{270}{4} = \boxed{67.5}$$

where does $f(x)=x^3-1$ cross x-axis?

Set $y=0$: $0=x^3-1$

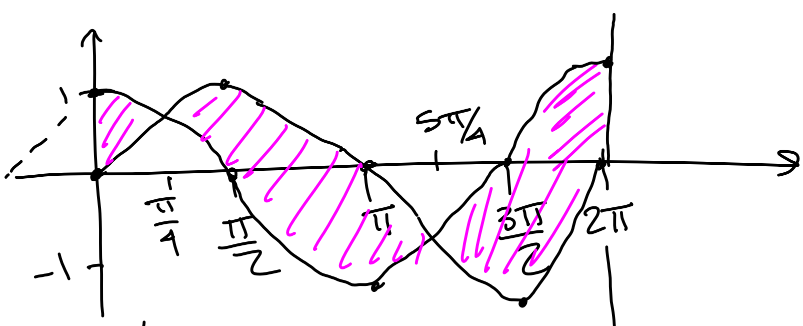
$1=x^3$

$x=1$

$A_1 = \frac{27}{4}$

$A_2 = \frac{243}{4}$

Example 2: Find the area of the region bounded by $y=\cos x$, $y=\sin x$, and the lines $x=0$, and $x=2\pi$.



Find intersection pts:

$\sin x = \cos x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$A_1 = \int_0^{\pi/4}\left[\cos x - \sin x\right]dx = (\sin x - (-\cos x))\Big|_0^{\pi/4} = (\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= \sin\frac{\pi}{4}+\cos\frac{\pi}{4}-\sin 0-\cos 0 = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-0-1 = \frac{2\sqrt{2}}{2}-1$$

$$= \sqrt{2}-1$$

$$A_2 = \int_{\pi/4}^{5\pi/4}\left[\sin x-\cos x\right]dx = \left|(\sin x+\cos x)\Big|_{\pi/4}^{5\pi/4}\right| = \left|\sin\left(\frac{5\pi}{4}\right)+\cos\left(\frac{5\pi}{4}\right) - \sin\pi/4 - \cos\pi/4\right|$$

$$= \left|-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right| = \left|-\frac{4\sqrt{2}}{2}\right| = 2\sqrt{2}$$

$$A_3 = \int_{5\pi/4}^{2\pi}\left[\cos x-\sin x\right]dx = (\sin x+\cos x)\Big|_{5\pi/4}^{2\pi} = \sin 2\pi + \cos 2\pi - \sin\frac{5\pi}{4} - \cos 5\pi/4$$

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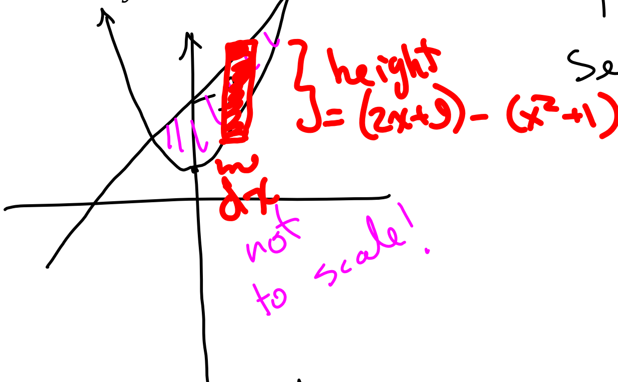
$$= 0 + 1 - \left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{-\sqrt{2}}{2}\right) = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 1 + \frac{2\sqrt{2}}{2}$$

$$= 1 + \sqrt{2}$$

Total Area $= A_1 + A_2 + A_3 = \sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2}$

$$= \boxed{4\sqrt{2}}$$

Example 3: Find the area of the region completely enclosed by the graphs of $y = x^2 + 1$ and $y = 2x + 9$.



Find the intersection points:

Set curves equal:

$$2x + 9 = x^2 + 1$$

$$0 = x^2 - 2x - 8$$

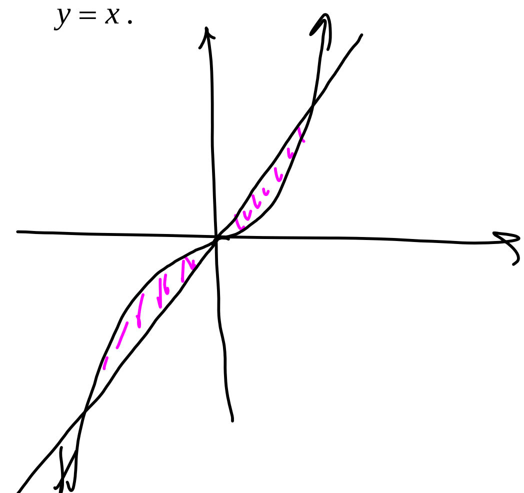
$$0 = (x - 4)(x + 2)$$

$$x = -2, \quad x = 4$$

$$\text{Area} = \int_{-2}^{4} \left((2x+9) - (x^2+1) \right) dx = \int_{-2}^{4} \left(-x^2 + 2x + 8 \right) dx$$

$$= -\frac{x^3}{3} + \frac{2x^2}{2} + 8x \Big|_{-2}^{4} = -\frac{4^3}{3} + 4^2 + 8(4) - \left(-\frac{(-2)^3}{3} + (-2)^2 + 8(-2) \right) = \boxed{36}$$

Example 4: Find the area of the region completely enclosed by the graphs of $y = x^3$ and $y = x$.



Find intersection pts:

Set $x^3 = x$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x+1)(x-1) = 0$$

$$x = 0, 1, -1$$

From symmetry, total area is

$$A = 2 \int_0^1 (x - x^3)\, dx$$

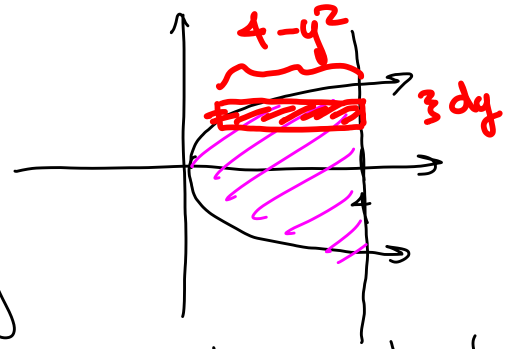
$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Big|_0^1 = 2\left[\frac{1^2}{2} - \frac{1^4}{4} - \left(\cancel{\frac{0^2}{2} - \frac{0^4}{4}}\right)\right] = 2\left[\frac{1}{2} - \frac{1}{4}\right]$$

$\cancel{} \to 0$

$$= 2\left[\frac{1}{4}\right] = \boxed{\frac{1}{2}}$$

Example 5: Find the area of the region completely enclosed by the graphs of $x = y^2$ and $x = 4$.

$$\int_{-2}^{2} (4 - y^2)\,dy$$



$$= 2\int_0^2 (4-y^2)\,dy \qquad [\text{from symmetry}]$$

$$= 2\left(4y - \frac{y^3}{3}\right)\Big|_0^2 = 2\left(4(2) - \frac{2^3}{3}\right) - 2\left(4(0) - \frac{0^3}{3}\right)$$

$$= 2\left(8 - \frac{8}{3}\right) - 0 = 2\left(\frac{24}{3} - \frac{8}{3}\right) = 2\left(\frac{16}{3}\right)$$

$$= \boxed{\frac{32}{3}}$$

Find intersection pts: Set x's equal:

$$y^2 = 4$$

$$y = \pm 2$$

Example 6: Find the area of the region completely enclosed by the graphs of $x = 3 - y^2$ and $x = y + 1$.

$$x = y + 1$$

$$y = x - 1$$

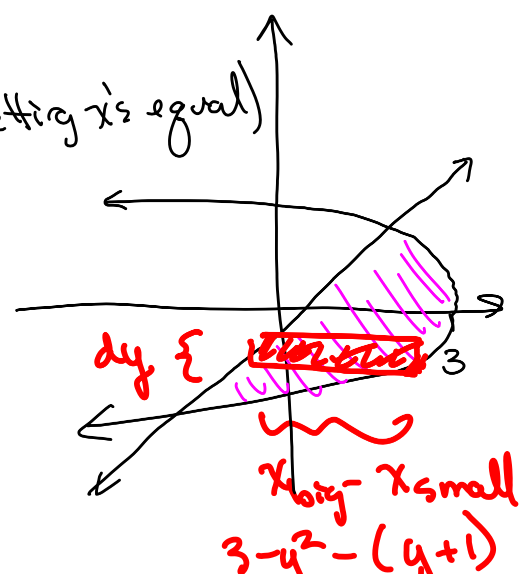
Find intersection Pts

$$3 - y^2 = y + 1 \quad (\text{setting x's equal})$$

$$0 = y^2 + y - 2$$

$$0 = (y+2)(y-1)$$

$$y = -2,\ y = 1$$



$$\text{Area} = \int_{-2}^{1} \left[3 - y^2 - (y+1)\right] dy$$

$$= \int_{-2}^{1} \left[3 - y^2 - y - 1\right] dy = \int_{-2}^{1} \left[2 - y^2 - y\right] dy = \left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]\Big|_{-2}^{1}$$

$$= \left[2(1) - \frac{1^3}{3} - \frac{1^2}{2}\right] - \left[2(-2) - \frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right] = \left[\frac{12}{6} - \frac{2}{6} - \frac{3}{6}\right] - \left[-4 + \frac{8}{3} - 2\right]$$

$$= \frac{7}{6} - \left[-6 + \frac{8}{3}\right] = \frac{7}{6} + \frac{36}{6} - \frac{16}{6} = \frac{43}{6} - \frac{16}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$$