1.2: Finding Limits Graphically and Numerically

Limit of a function:

Definition of a Limit:

$$\lim_{x \to a} f(x) = L$$

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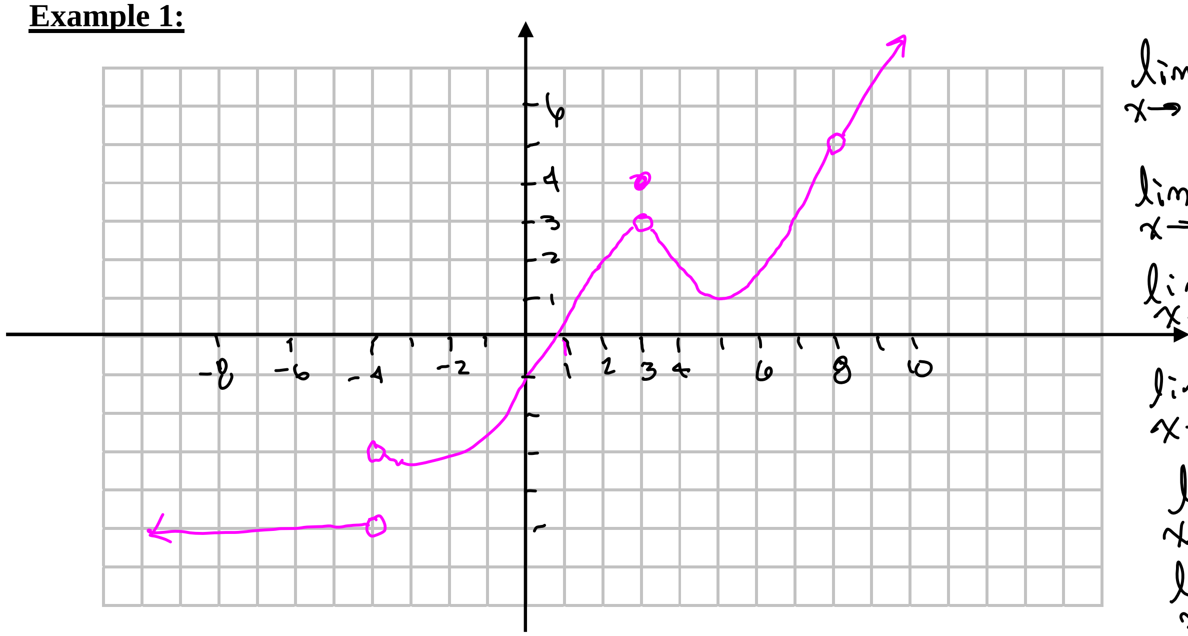
The statement above means that we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

We read this as "the limit of $f(x)$, as x approaches a, is equal to L."

Alternative notation: $f(x) \to L$ as $x \to a$. ($f(x)$ approaches L as x approaches a)

Finding limits from a graph:

Example 1:



$\lim_{x \to 2} f(x) = 2$

$\lim_{x \to 3} f(x) = 3$

$\lim_{x \to 5} f(x) = 1$

$\lim_{x \to -4} f(x)$ does not exist

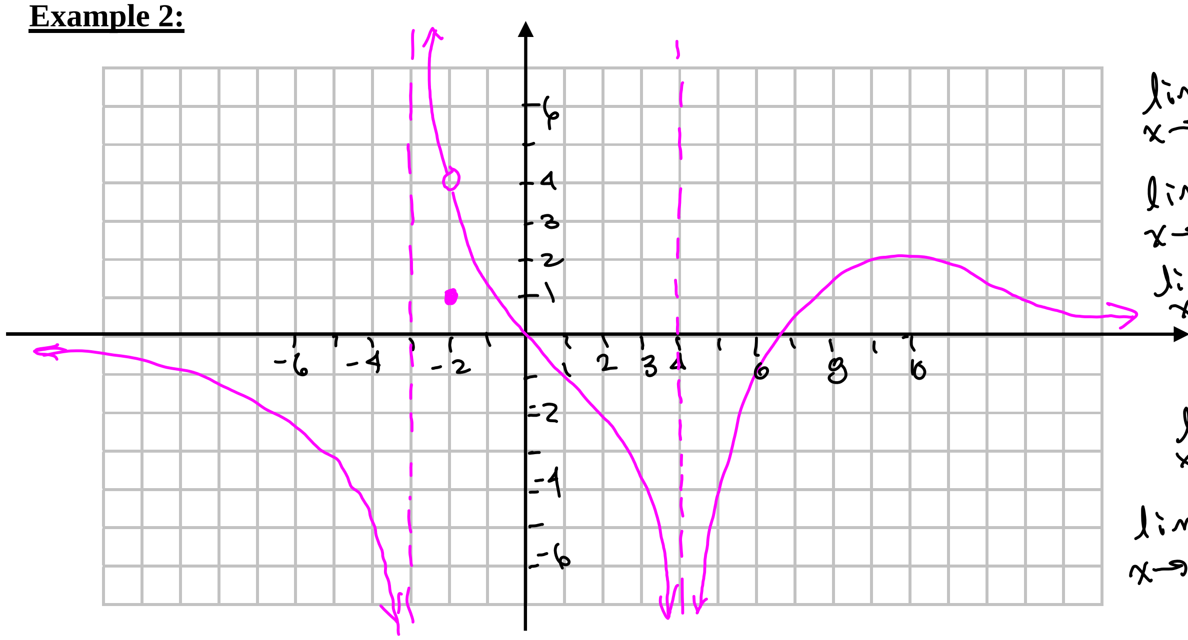
$\lim_{x \to 7} f(x) = 3$

$\lim_{x \to 8} f(x) = 5$

Note: $f(3) = 4$

$f(6) \approx 1.5$

Example 2:



$\lim_{x\to -3} f(x)$ does not exist

$\lim_{x\to -2} f(x) = 4$

$\lim_{x\to 4} f(x)$ does not exist, but we write $\lim_{x\to 4} f(x) = -\infty$

$\lim_{x\to 8} f(x) \approx 1.4$

$\lim_{x\to \infty} f(x) = 0$

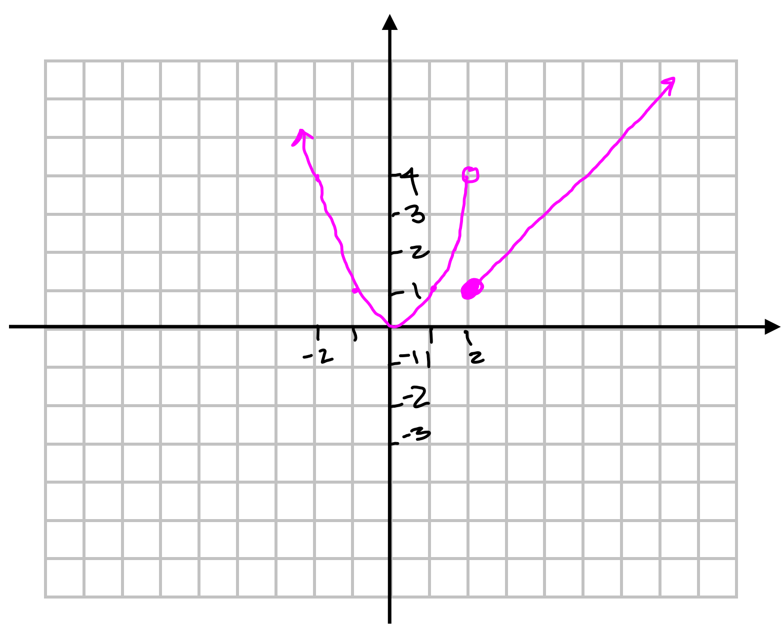
$\lim_{x\to -\infty} f(x) = 0$

Example 3: Graph the function. Use the graph to determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to -1} f(x)$.

$$f(x)=\begin{cases} x-1 & \text{if } x\geq 2 \\ x^2 & \text{if } x<2 \end{cases}$$

$\lim_{x\to 2} f(x)$ does not exist

$\lim_{x\to -1} f(x) = 1$



Ex $3\frac{1}{2}$: $f(x)=\begin{cases} x+b & \text{if } x\geq 2 \\ x^2 & \text{if } x<2 \end{cases}$

Find the value of b that will cause $\lim_{x\to 2} f(x)$ to exist.

$y = x+b = x^2$ for $x=2$

$2+b = 2^2$

$2+b = 4 \Rightarrow b = 2$

Function would be $f(x)=\begin{cases} x+2 & x\geq 2 \\ x^2 & x<2 \end{cases}$

Then $\lim_{x\to 2} f(x) = 4$

Finding limits numerically:

<u>Example 4:</u> For the function $f(x)=\dfrac{x-5}{x^2-25}$, make a table of function values corresponding to values of x near 5.

Use the table to estimate the value of $\lim\limits_{x\to 5}\dfrac{x-5}{x^2-25}$.

x	$f(x)=\dfrac{x-5}{x^2-25}$
4.8	0.10204
4.9	0.10101
4.95	0.1005
4.99	0.1001
4.999	0.10001

5.0001	0.0999900001
5.001	0.09999
5.01	0.099
5.05	0.0995
5.1	0.09901
5.2	0.09904

From table,

$$\lim_{x\to 5}\frac{x-5}{x^2-25}=0.1$$

<u>Example 5:</u> Make a table of values and use it to estimate $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$.

This limit does not exist.

<u>Common reasons</u> $\lim_{x\to c} f(x)$ <u>may not exist:</u>

1. $f(x)$ approaches a different value when approached from the left of *c*, compared to when approached from the right of *c*.

2. $f(x)$ increases or decreases without bound as *x* approaches *c*.

3. $f(x)$ oscillates between two values as *x* approaches *c*.

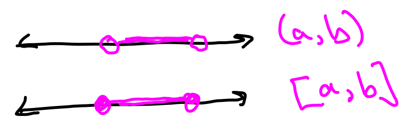
ϵ : epsilon (lower-case)
δ : delta (lower-case)

The formal (epsilon-delta) definition of a limit:

<u>Definition</u>:

Δ : upper-case delta

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then



$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

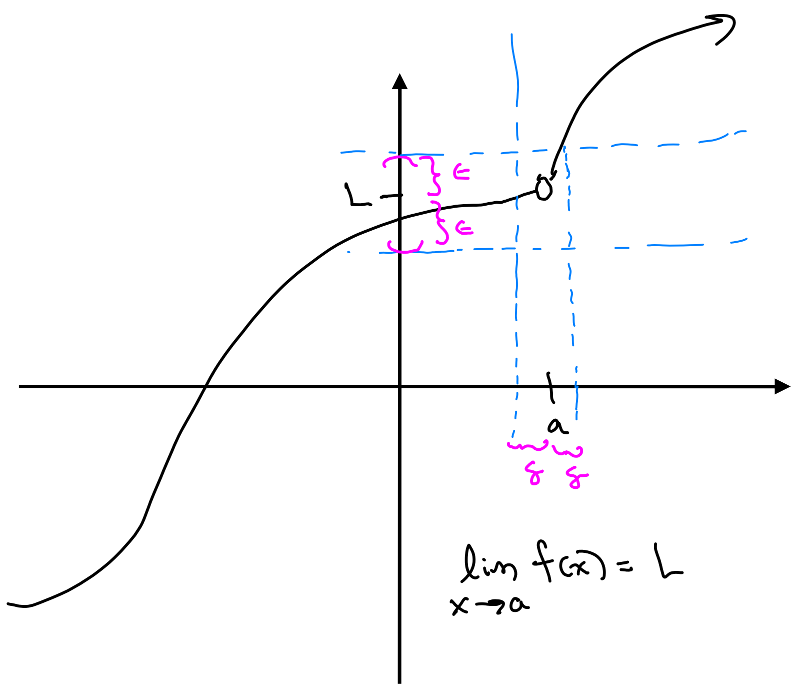
$0 < |x-a| < \delta \implies |f(x)-L| < \epsilon$

$|f(x)-L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

} equivalent

if x is within δ of a, then $f(x)$ is within ϵ of L.

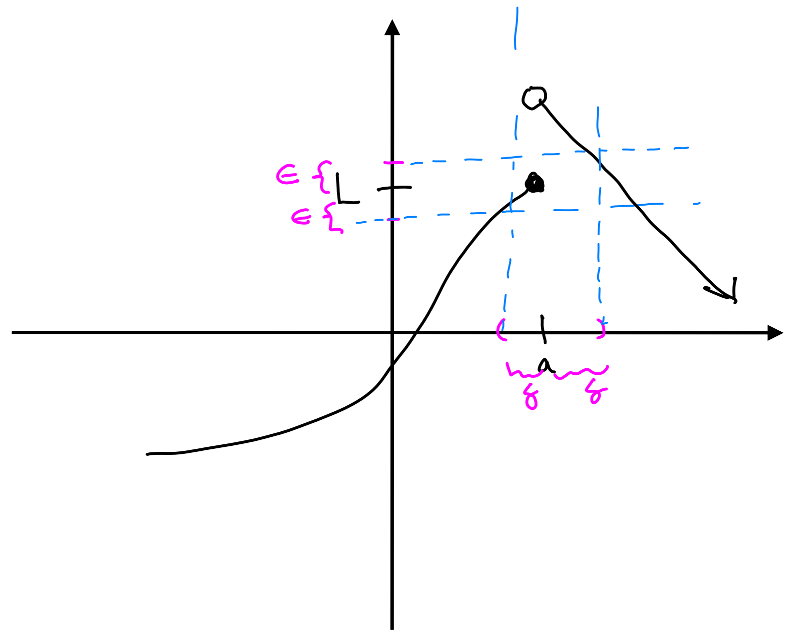
<u>Example 6:</u>



$\lim_{x\to a} f(x) = L$

Example 7:

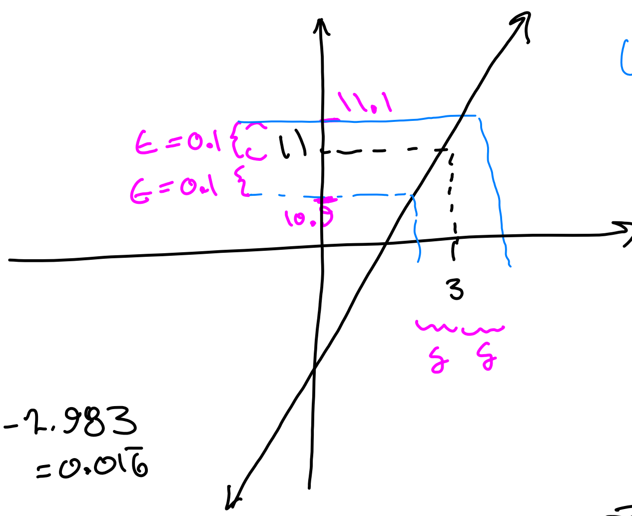
For this δ, the x's to the right of a (but within δ of a) will produce $f(x\text{'s})$ that fall outside the ε-window.



Example 8: How close to 3 must we take x so that $6x-7$ is within 0.1 of 11?

$f(x) = 6x - 7$

$f(3) = 6(3) - 7 = 11$



(not to scale)

Put 10.9 into $f(x) = 6x - 7$ in place of $f(x)$:

$10.9 = 6x - 7$

$17.9 = 6x$

$x = \frac{17.9}{6} \approx 2.98\bar{3} \Rightarrow 3 - 2.983 = 0.01\bar{6}$

Put in 11.1 for $f(x)$: $11.1 = 6x - 7$

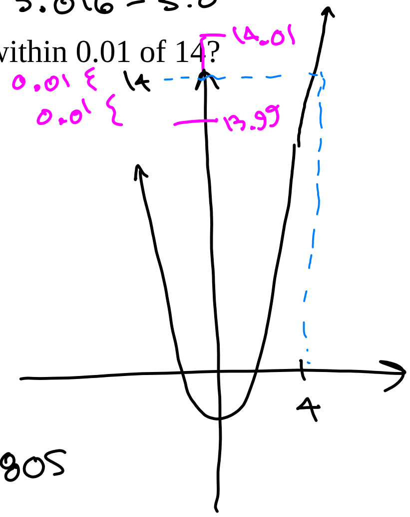
$18.1 = 6x$

$x = \frac{18.1}{6} \approx 3.01\bar{6} \Rightarrow \delta = 3.01\bar{6} - 3.0 = 0.01\bar{6} \Rightarrow$ choose $\boxed{\delta < 0.01\bar{6}}$

Example 9: How close to 4 must we take x so that x^2-2 is within 0.01 of 14?

$f(x) = x^2 - 2$

$f(4) = 4^2 - 2 = 14$



$x^2 - 2 = 14.01$

$x^2 = 16.01$

$x \approx 4.001249805$

$|4 - x| \approx 0.001249805$

more restrictive

$x^2 - 2 = 13.99$

$x^2 = 15.99$

$x \approx 3.998749805$

$|x - 4| = 4 - 3.998749805$

≈ 0.0012501954

$\boxed{\text{Choose } \delta < 0.001249805}$

$\delta = 0.001$ would work

Example 10: Prove that $\lim_{x \to 4}(2x-5)=3$ using the definition of a limit.

Scratchwork

Let $\epsilon > 0$.
(Take any positive ϵ.)

we want
$|f(x)-L| < \epsilon$ when $0<|x-4|<\delta$

Here, $f(x) = 2x-5$
$L = 3$

$|f(x)-L| = |(2x-5)-3| < \epsilon$

$|2x-8| < \epsilon$

$|2(x-4)| < \epsilon$

$|2||x-4| < \epsilon$

$2|x-4| < \epsilon$

$|x-4| < \frac{\epsilon}{2}$

Let $\delta = \frac{\epsilon}{2}$

Note: $|AB| = |A||B|$

Proof: Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{2}$.

Assume that $0 < |x-4| < \delta$.

(We will now show that $|f(x)-3| < \epsilon$.)

$$|(2x-5)-3| = |2x-8| = |2(x-4)| = |2||x-4|$$

$$= 2|x-4| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

because $|x-4|<\delta$

$\therefore \lim_{x \to 4}(2x-5) = 3$.

$\therefore$ means "therefore"

Example 11: Prove that $\lim_{x \to -3}(5x+1) = -14$ using the definition of a limit.

$$\lim_{x \to -3}(5x+1) = -14$$

Everyone did this on their own during class.

For my proof, see Archives: 2413 Sum 2014.

Example 12: Prove that $\lim_{x \to 3} x^2 = 9$ using the definition of a limit.

You aren't responsible for $\epsilon - \delta$ proofs involving quadratic function.

To see the proof, Go to Archives: 2413 Sum 2014.

Example 13: Prove that $\lim_{x \to 2}(x^2 - x + 6) = 8$ using the definition of a limit.

You aren't responsible for $\epsilon - \delta$ proofs involving quadratic function.

To see the proof, Go to Archives: 2413 Sum 2014.