1.3: Evaluating Limits Analytically

Some basic limits:

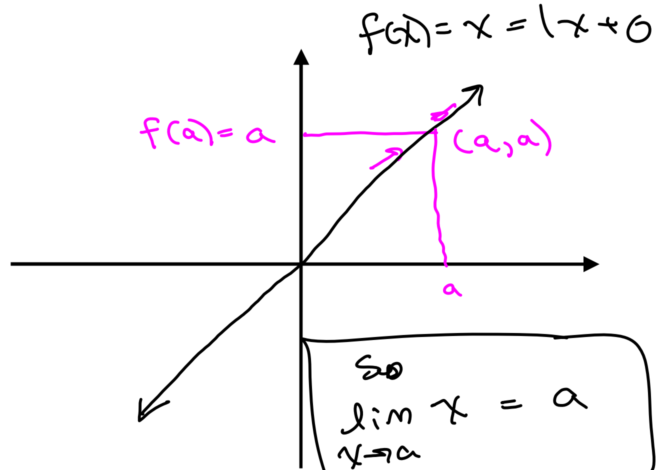
Example 1: Determine $\lim_{x \to a} x$.

$\lim_{x \to a} x$

Write as a function: $f(x) = x$

x	$f(x) = x$
$a + 0.01$	$a + 0.01$
$a + 0.001$	$a + 0.001$
$a + 0.0001$	$a + 0.0001$

↓ approach a



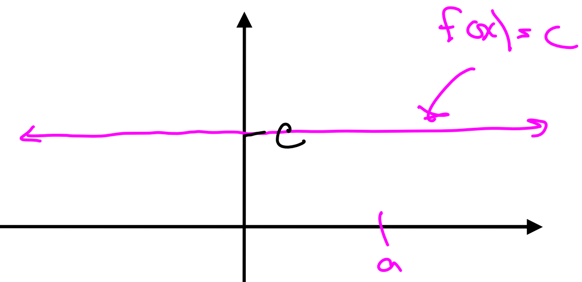
so $\lim_{x \to a} x = a$

Example 2: Determine $\lim_{x \to a} c$.

$\lim_{x \to a} C = ?$

(c is a constant)

x	$f(x) = c$
$a + 0.1$	c
$a + 0.001$	c
$a + 0.0001$	c
	approach c



$\lim_{x \to a} C = C$

Laws (or properties) of limits:

Limit Laws:

Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ if c is a constant

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^{n/p} = [\lim_{x\to a} f(x)]^{n/p}$, if n and p are integers with no common factor, $p \neq 0$, and provided that $[\lim_{x\to a} f(x)]^{n/p}$ is a real number.

Combining the facts that $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, and the limit laws shows that $\lim_{x\to a} x^n = a^n$ where n is a positive integer. This lets us use direct substitution for evaluating limits of polynomials.

Direct Substitution Property:

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

Example 3: Determine $\lim_{x\to 3}(4x^2-2x+1)$.

$$\lim_{x\to 3}(4x^2-2x+1) = 4(3)^2-2(3)+1 = 36-6+1 = \boxed{31}$$

Example 4: Determine $\lim_{x\to -2}\dfrac{2x^2-6x+5}{x-3}$.

$$\lim_{x\to -2}\frac{2x^2-6x+5}{x-3} = \frac{2(-2)^2-6(-2)+5}{-2-3} = \frac{8+12+5}{-5} = \frac{25}{-5} = \boxed{-5}$$

Example 5: Determine $\lim\limits_{x\to 2}\sqrt[3]{4x-x^4}$.

$$\lim_{x\to 2}\sqrt[3]{4x-x^4} = \sqrt[3]{4(2)-2^4} = \sqrt[3]{8-16} = \sqrt[3]{-8} = \boxed{-2}$$

Limit Law #6 (used in the previous example) is a special case of the following theorem:

> Limit of a Composite Function
>
> If f and g are functions such that $\lim\limits_{x\to a} g(x) = L$ and $\lim\limits_{x\to L} f(x) = f(L)$
>
> $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(L)$

Example 6: Determine $\lim\limits_{x\to 4}\dfrac{x^2+x-20}{x^2-7x+12}$.

Direct substitution:

$$\frac{4^2+4-20}{4^2-7(4)+12} \Rightarrow \frac{0}{16-28+12} \Rightarrow \frac{0}{0}$$

not defined!

This is called an indeterminate form.

Limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, ∞^0, and several others are called indeterminate forms (More on this in Calc II)

$$\lim_{x\to 4}\frac{x^2+x-20}{x^2-7x+12} = \lim_{x\to 4}\frac{(x+5)\cancel{(x-4)}}{\cancel{(x-4)}(x-3)}$$

$$= \lim_{x\to 4}\frac{x+5}{x-3}$$

$$= \frac{4+5}{4-3} = \frac{9}{1} = \boxed{9}$$

Example 7: Determine $\lim\limits_{x\to -2}\dfrac{\sqrt{x+3}-1}{x+2}$.

Direct Sub

$$\frac{\sqrt{-2+3}-1}{-2+2} \Rightarrow \frac{0}{0}$$

indeterminate

$$\lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2}$$

$$= \lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2}\left(\frac{\sqrt{x+3}+1}{\sqrt{x+3}+1}\right)$$

$$= \lim_{x\to -2}\frac{(\sqrt{x+3})^2-1^2}{(x+2)(\sqrt{x+3}+1)}$$

$$= \lim_{x\to -2}\frac{x+3-1}{(x+2)(\sqrt{x+3}+1)}$$

$$= \lim_{x\to -2}\frac{\cancel{x+2}}{\cancel{(x+2)}(\sqrt{x+3}+1)} = \lim_{x\to -2}\frac{1}{\sqrt{x+3}+1}$$

$$= \frac{1}{\sqrt{-2+3}+1} = \frac{1}{\sqrt{1}+1} = \boxed{\frac{1}{2}}$$

Example 8: Determine $\lim_{x\to 3}\frac{x+4}{x-3}$.

Direct substitution:

$$\frac{3+4}{3-3} \Rightarrow \frac{7}{0} \text{ not indeterminate}$$

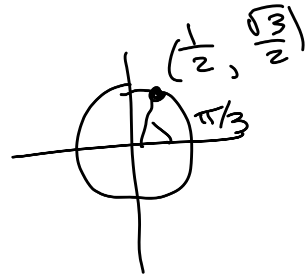
Skip for now. This limit does not exist.

This limit does not exist. More on this in a later section.

Limits of trigonometric functions:

These can also be evaluated through direct substitution, thanks to the two limits below, along with the limit laws.

$$\lim_{x\to a}\sin x = \sin a \qquad \lim_{x\to a}\cos x = \cos a$$



Example 9: Determine $\lim_{x\to \frac{\pi}{3}}\tan x$.

$$\lim_{x\to \frac{\pi}{3}}(\tan x) = \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2}$$

$$= \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$$

Example 10: Determine $\lim_{x\to \frac{\pi}{4}}\frac{\cos x - \sin x}{\cos(2x)}$

Double-angle formula

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Direct sub

$$\frac{\cos\frac{\pi}{4} - \sin\frac{\pi}{4}}{\cos\frac{2\pi}{4}}$$

$$\frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{0} \Rightarrow \frac{0}{0}$$

indeterminate.

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x - \sin x}{\cos(2x)} = \lim_{x\to\frac{\pi}{4}}\frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\cancel{\cos x - \sin x}}{\cancel{(\cos x - \sin x)}(\cos x + \sin x)}$$

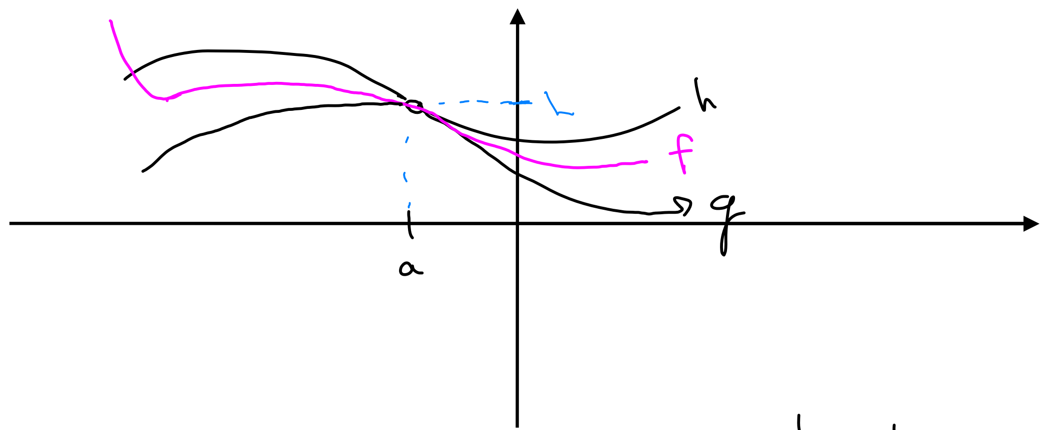
$$= \lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x + \sin x} = \frac{1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}$$

$$= \frac{1}{\frac{2\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$

The Squeeze (or Sandwich or Pinching) Theorem:

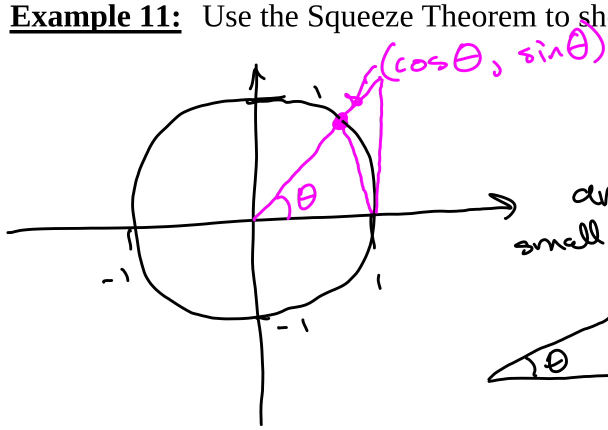
If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing near ~~x~~ a, except possibly at ~~x~~ a itself. If $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$, then

$\cancel{\lim\limits_{x\to a} h(x) = L}$. $\quad \lim\limits_{x\to a} f(x) = L$



Direct sub: $\dfrac{\sin 0}{0} \Rightarrow \dfrac{0}{0}$ indeterminate form

Example 11: Use the Squeeze Theorem to show that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.



area of small triangle $\le$ area of circle sector $\le$ area of big triangle

This proof was done on the whiteboard. To see it, go to Archives: Summer 2016 2413

Two important limits: ★ ★ ★ know these!

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=0$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \left(\frac{1-\cos x}{x}\right) = 0$

Direct Sub: $\frac{1-\cos 0}{0} \Rightarrow \frac{1-1}{0} \Rightarrow \frac{0}{0}$ indeterminate

$= \frac{1}{1} = 1$

Note: This means that $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1$, $\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0$, and $\lim_{x\to 0}\left(\frac{x}{1-\cos x}\right)$ does not exist.

$\lim_{x\to 0}(-1)\left(\frac{1-\cos x}{x}\right) = -1\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)$

$= -1(0) = 0$

Example 12: $\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right)$

$\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{x}\right)(\tan x) = \lim_{x\to 0}\left(\frac{\cancel{\cos x}}{x}\right)\left(\frac{\sin x}{\cancel{\cos x}}\right)$

$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \boxed{1}$

Example 13: $\lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{\cot x}\right)$

$\lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{\cot x}\right) = \lim_{x\to \pi/2}\left(\frac{\cos x}{\frac{\cos x}{\sin x}}\right) = \lim_{x\to \pi/2}\left(\cancel{\cos x}\cdot\frac{\sin x}{\cancel{\cos x}}\right)$

$= \lim_{x\to \frac{\pi}{2}}(\sin x) = \sin\frac{\pi}{2} = \boxed{1}$

Direct Sub

$\frac{\cos\frac{\pi}{2}}{\frac{\cos \pi/2}{\sin \pi/2}} \Rightarrow \frac{0}{0/1} \Rightarrow \frac{0}{0}$ indeterminate

Example 14: $\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)$

$\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right) = \lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)\left(\frac{5}{5}\right)$

$= \frac{5}{1}\lim_{x\to 0}\left(\frac{1-\cos(5x)}{5x}\right)$

$= 5\lim_{5x\to 0}\left(\frac{1-\cos(5x)}{5x}\right) = 5(0) = \boxed{0}$

We know $\lim_{u\to 0}\left(\frac{1-\cos u}{u}\right) = 0$

these must match

Note:

As $x\to 0$, $5x\to 0$ also

If $\lim_{x\to 0} x = 0$, then $\lim_{x\to 0} 5x = 5\lim_{x\to 0} x = 5(0) = 0$

Know $\lim_{x\to 0} \frac{\sin x}{x} = 1$

must match

Example 15: $\lim_{x\to 0}\left(\frac{\sin(3x)}{7x}\right)$

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{7x}\right) = \frac{1}{7}\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)$$

$$= \frac{1}{7}\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)\left(\frac{3}{3}\right) = \frac{1}{7}(3)\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right) = \frac{3}{7}\lim_{3x\to 0}\left(\frac{\sin 3x}{3x}\right)$$

(Because $3x \to 0$ as $x \to 0$)

$$= \frac{3}{7}(1) = \boxed{\frac{3}{7}}$$

Example 16: $\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)}$

$$= \lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)}\left(\frac{x}{x}\right) = \lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)\left(\frac{x}{\sin 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)\left(\frac{3}{3}\right)\cdot\lim_{x\to 0}\left(\frac{x}{\sin(2x)}\right)\left(\frac{2}{2}\right) = 3\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\cdot\frac{1}{2}\lim_{x\to 0}\left(\frac{2x}{\sin 2x}\right)$$

$$= 3\lim_{3x\to 0}\left(\frac{\sin 3x}{3x}\right)\cdot\frac{1}{2}\lim_{2x\to 0}\left(\frac{2x}{\sin 2x}\right)$$

[Because as $x\to 0$, $3x\to 0$ and $2x \to 0$ also]

$$= 3(1)\left(\frac{1}{2}\right)(1) = \boxed{\frac{3}{2}}$$

Example 17: $\lim_{x\to 0}\frac{x^3}{\sin^3(4x)}$

$$= \lim_{x\to 0}\left(\frac{x}{\sin(4x)}\right)^3 = \lim_{x\to 0}\left(\frac{x}{\sin(4x)}\cdot\frac{4}{4}\right)^3$$

$$= \lim_{x\to 0}\left(\frac{4x}{\sin 4x}\cdot\frac{1}{4}\right)^3 = \left[\lim_{x\to 0}\left(\frac{4x}{\sin 4x}\right)\cdot\frac{1}{4}\right]^3 = \left[\lim_{x\to 0}\left(\frac{4x}{\sin 4x}\right)\right]^3\cdot\left[\frac{1}{4}\right]^3$$

$$= \left[\lim_{4x\to 0}\left(\frac{4x}{\sin 4x}\right)\right]^3\cdot\frac{1}{64}$$

(because $4x\to 0$ as $x\to 0$)

$$= 1^3\cdot\frac{1}{64} = \boxed{\frac{1}{64}}$$

Example 18: $\lim_{x\to 0}\frac{\cot 3x}{\csc 8x}$

$$\lim_{x\to 0}\left[\frac{\cot(3x)}{\csc(8x)}\right] = \lim_{x\to 0}\left[\cot(3x)\cdot\frac{1}{\csc(8x)}\right] = \lim_{x\to 0}\left[\frac{\cos 3x}{\sin 3x}\cdot\frac{1}{\frac{1}{\sin 8x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cos 3x}{\sin 3x}\cdot\sin 8x\right] = \lim_{x\to 0}\left[\cos 3x\cdot\frac{1}{\sin 3x}\cdot\sin 8x\cdot\frac{x}{x}\right]$$

$$= \lim_{x\to 0}\left[\cos 3x\right]\cdot\lim_{x\to 0}\left[\frac{x}{\sin 3x}\right]\cdot\lim_{x\to 0}\left[\frac{\sin 8x}{x}\right]$$

See next page

Ex 18 cont'd:

$$= \cos(3(0)) \cdot \lim_{x \to 0}\left[\frac{x}{\sin 3x}\left(\frac{3}{3}\right)\right] \cdot \lim_{x \to 0}\left[\frac{\sin 8x}{x}\left(\frac{8}{8}\right)\right]$$

$$= \cos(0) \cdot \frac{1}{3}\lim_{x \to 0}\left[\frac{3x}{\sin 3x}\right] \cdot \frac{8}{1} \cdot \lim_{x \to 0}\left[\frac{\sin 8x}{8x}\right]$$

$$= 1 \cdot \frac{1}{3} \cdot \lim_{3x \to 0}\left[\frac{3x}{\sin 3x}\right] \cdot 8 \cdot \lim_{x \to 8x}\left[\frac{\sin 8x}{8x}\right]$$

(because if $x \to 0$, then $8x \to 0$ and $3x \to 0$ also)

$$= 1 \cdot \frac{1}{3} \cdot 1 \cdot 8 \cdot 1 = \boxed{\frac{8}{3}}$$