1.4: Continuity and One-Sided Limits

One-Sided Limits:

$\lim_{x\to a^-} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the left.

$\lim_{x\to a^-} f(x) = L$

$\lim_{x\to a^+} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the right.

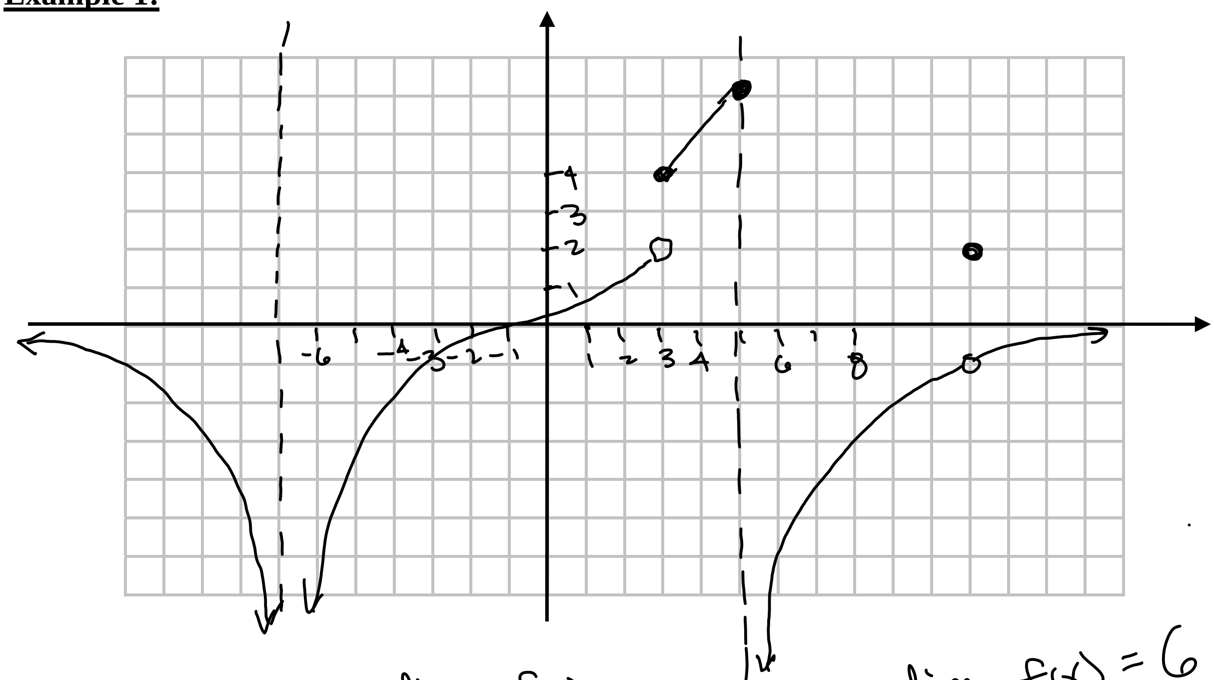
$\lim_{x\to a^+} f(x) = L$

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

iff = if and only if

$\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Example 1:



$\lim_{x\to -6^-} f(x) = -7$

$\lim_{x\to -7^+} f(x) = -\infty$

$\lim_{x\to -7^-} f(x) = -\infty$

$\lim_{x\to -7} f(x) = -\infty$

none of these 3 limits exists

$\lim_{x\to -4^-} f(x) = -2$

$\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^+} f(x) = 4$

$\lim_{x\to 3} f(x)$ does not exist

$\lim_{x\to 5^-} f(x) = 6$

$\lim_{x\to 5^+} f(x) = -\infty$ (does not exist)

$\lim_{x\to 5} f(x)$ does not exist

$\lim_{x\to 11} f(x) = -1$

Note that $f(11) = 2$

Example 2: Determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$. , and $\lim_{x\to 2} f(x)$

$$f(x)=\begin{cases} x^2 & \text{if } x\le 1 \\ x-3 & \text{if } x>1 \end{cases}.$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2) = 1^2 = \boxed{1}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-3) = 1-3 = \boxed{-2}$

$\lim_{x\to 1} f(x)$ $\boxed{\text{does not exist}}$ (because the left-side and right-side limits don't match)

$\lim_{x\to 2} f(x) = \lim_{x\to 2} (x-3) = 2-3 = \boxed{-1}$

Example 3: Determine $\lim_{x\to -2^-} f(x)$, $\lim_{x\to -2^+} f(x)$, and $\lim_{x\to -2} f(x)$.

$1-x^3$

$$f(x)=\begin{cases} 1-x^3 & \text{if } x\le -2 \\ 7-x & \text{if } x>-2 \end{cases}.$$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (1-x^3) = 1-(-2)^3 = 1-(-8) = 1+8 = \boxed{9}$

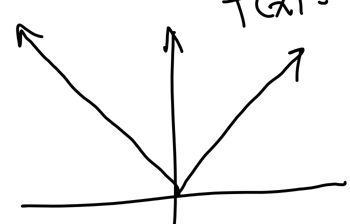
$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (7-x) = 7-(-2) = \boxed{9}$

$\lim_{x\to 2} f(x) = 7-2 = \boxed{5}$ $\quad \lim_{x\to -2} f(x) = \boxed{9}$

$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} (1-x^3) = 1-(-5)^3 = 1-(-125) = 1+125 = \boxed{126}$

Example 4: Determine $\lim_{x\to 0} |x|$, if it exists.

$f(x) = |x|$



$$f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

From graph, $\lim_{x\to 0} |x| = 0$

$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = -0 = 0$

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} (x) = 0$

Thus $\lim_{x\to 0} |x| = \boxed{0}$

Example 5: Determine $\lim_{x\to 1} f(x)$ and $\lim_{x\to -2} f(x)$, where $f(x)=\begin{cases}\sqrt{5-x} & \text{if } x\ge 1\\ 3x^2-7 & \text{if } x<1\end{cases}$.

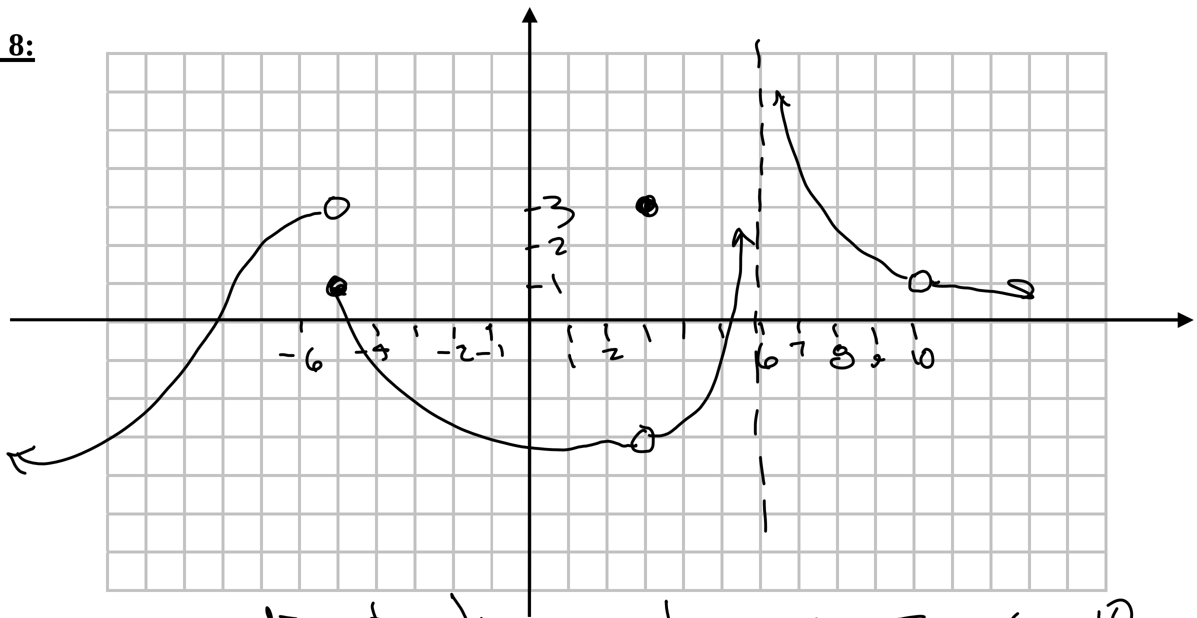
Example 6: Determine $\lim_{x\to 0} f(x)$, where $f(x)=\begin{cases}\cos x & \text{if } x<0\\ 2 & \text{if } x=0\\ \sqrt{x+1} & \text{if } x>0\end{cases}$.

Example 7: Determine $\lim_{x\to 1}\frac{|x-1|}{x-1}$.

Continuity of a function:

In most cases, we can think of a continuous function as one that can be drawn "without lifting your pencil from the paper". In other words, there are no holes, breaks, or jumps.

Example 8:



This function has discontinuities at $-5, 3, 6, 10$

Continuous on intervals $(-\infty, -5), (-5, 3), (3, 6), (6, 10), (10, \infty)$

Definition: A function f is continuous at a number a, an interior point of its domain, if

$$\lim_{x \to a} f(x) = f(a)$$

Conditions for Continuity:

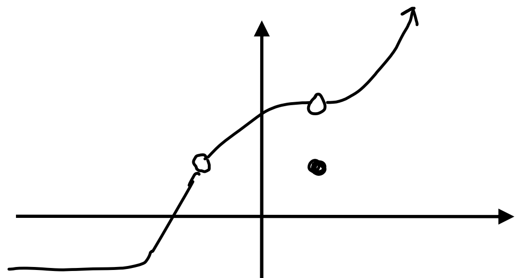
In order for f to be continuous at a, all the following conditions must hold:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Types of discontinuities:

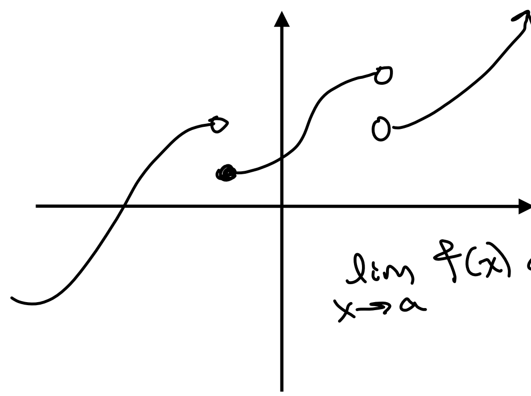
1. Removable discontinuity
2. Infinite discontinuity
3. Jump discontinuity
4. Oscillating discontinuity

What do these look like? For each type of discontinuity, what condition of continuity is violated?



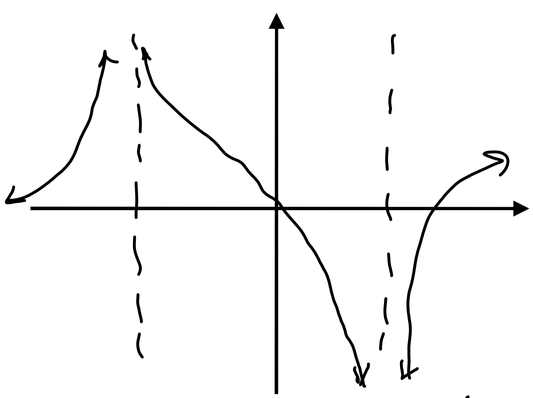
$\lim_{x\to a} f(x)$ exists, but either $f(a)$ does not exist, or $f(a) \neq \lim_{x\to a} f(x)$

Removable Discontinuity



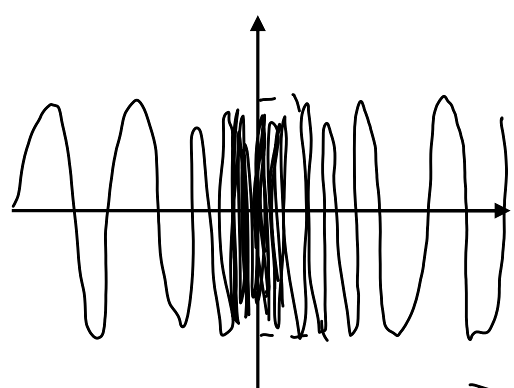
$\lim_{x\to a} f(x)$ does not exist

Jump Discontinuity



Infinite discontinuity

$\lim_{x\to a} f(x)$ does not exist because $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ (or both)



Oscillating Discontinuity

$\lim_{x\to a} f(x)$ does not exist (and is not $\pm\infty$, and the one-sided limits don't exist either)

Continuity at an endpoint of the domain:

A function *f* is <u>continuous at a number *a*</u>, a left endpoint of its domain, if $\lim_{x \to a^+} f(x) = f(a)$.

A function *f* is <u>continuous at a number *a*</u>, a right endpoint of its domain, if $\lim_{x \to a^-} f(x) = f(a)$.

<u>Example 9:</u> $f(x) = \sqrt{4 - x}$

$y = \sqrt{4 - x}$

$y = \sqrt{-(x-4)}$

where is $4 - x = 0$?

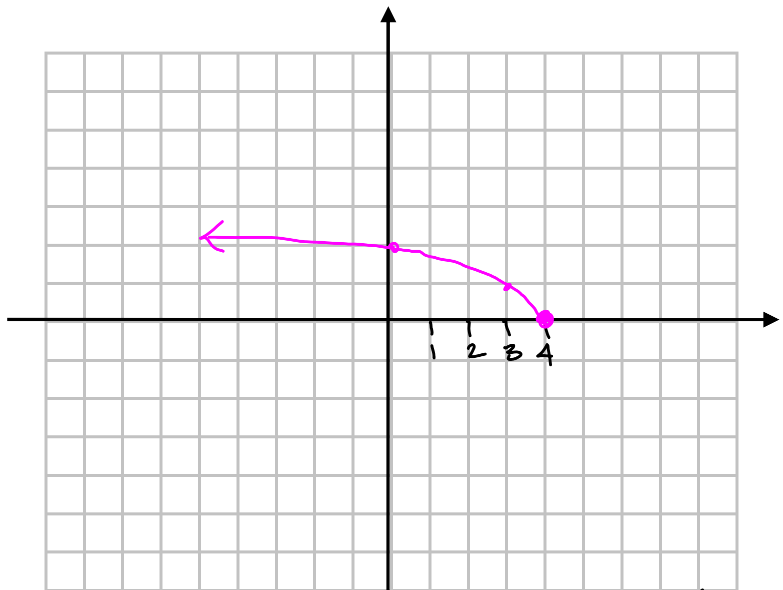
$x = 4$

Domain: $(-\infty, 4]$

Check: $4 - x \geq 0$

$-x \geq -4$

$x \leq 4$ ✓



$f(x) = \sqrt{4-x}$ is continuous on $(-\infty, 4]$. At 4, we have $\lim_{x \to 4^-} f(x) = f(4) = 0$

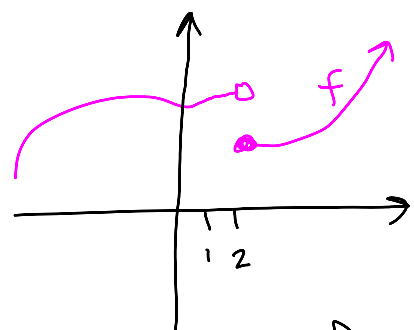
One-sided continuity:

<u>Definition</u>: A function *f* is <u>continuous from the left</u> at a number *a* if

$$\lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a)$$

A function *f* is <u>continuous from the right</u> at a number *a* if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$



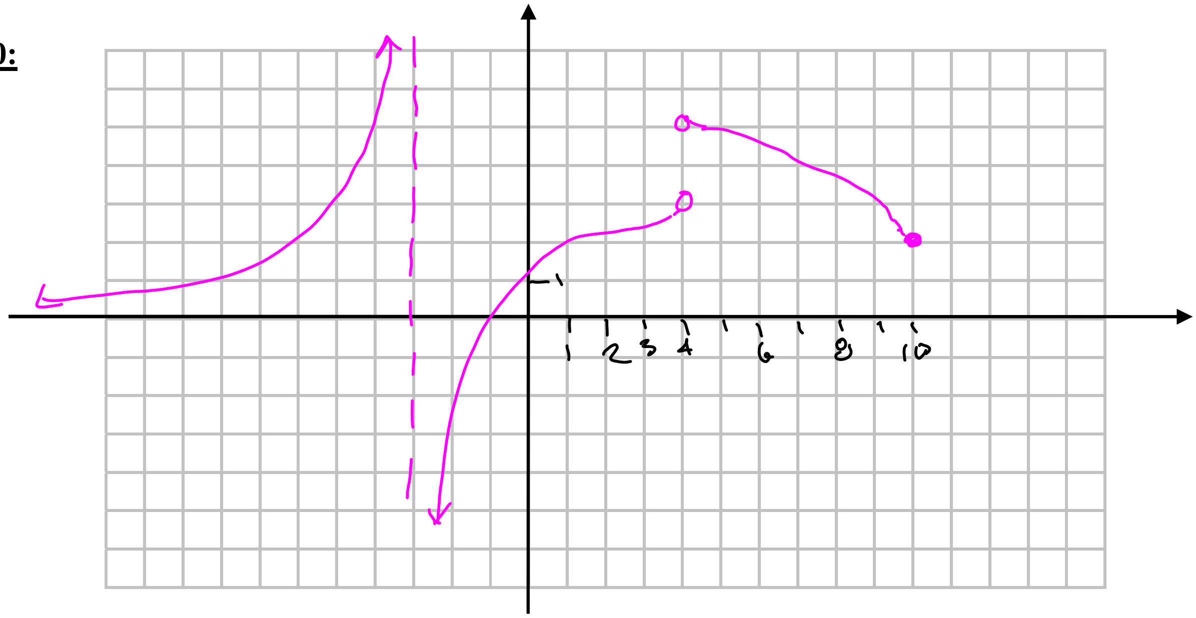
f is continuous from the right at 2, because $\lim_{x \to 2^+} f(x) = f(2)$

Continuity on an interval:

<u>Definition</u>: A function *f* is continuous on an interval if it is continuous at every point in the interval.

(If *f* is defined only at one side of an endpoint, then only continuity from the left or right is needed for it to be continuous at the endpoint.)

Example 10:



On what intervals is the above function continuous?

f is continuous on $(-\infty, -3)$, $(-3, 4)$, $(4, 10]$

Theorem: If *f* and *g* are continuous at *a* and *c* is a constant, then $f+g$, $f-g$, fg, cf are also continuous at *a*. The quotient $\frac{f}{g}$ is also continuous at *a* if $g(a) \neq 0$.

Theorem:
Polynomials, rational functions, root functions, and trigonometric functions are continuous at every number in their domains.

Theorem:
If f is continuous at $b = \lim_{x \to a} g(x)$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Theorem:
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 11: Where is $f(x) = \sin\left(\frac{1}{x}\right)$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Oscillating discontinuity at 0

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist

Example 12: Where is $f(x) = \sqrt{\frac{x^2+1}{(x-1)^3}}$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Note: x^2+1 is always positive (must be at least 1)

For this function to be defined, the denominator must be positive: $x - 1 > 0$

$x > 1$

Domain: $(1, \infty)$

Continuous on $(1, \infty)$

Example 13: Determine the values of *x*, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{5x^3 - 8x^2}{x - 7}$$

Undefined for $x = 7$.

Domain: $(-\infty, 7) \cup (7, \infty)$

Infinite discontinuity at 7.

Note that $\lim_{x \to 7} f(x)$ does not exist

Example 14: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{x^3 - 3x^2}{x^2 + 4}$$

Note: x^2+4 is never 0.
Can't be negative
Smallest x^2+4 can be is 4

Domain: $(-\infty, \infty)$

$\boxed{\text{Continuous on } (-\infty, \infty)}$

Example 15: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} x+3 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \\ x^2+3 & \text{if } x < 0 \end{cases}.$$

Note that $y = x+3$ and $y = x^2+3$ are continuous everywhere. So only possible discontinuity is where the segments join.

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x^2+3) = 0^2+3 = 3$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (x+3) = 0+3 = 3$

Therefore $\lim\limits_{x\to 0} f(x) = 3$. However, $f(0) = 4$. So f is discontinuous at 0, because $f(0) \neq \lim\limits_{x\to 0} f(x)$.
(removable discontinuity at 0)

Example 16: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 5 & \text{if } x = 1 \\ x+5 & \text{if } x < 1 \end{cases}.$$

Again, the segments $y = 5x$ and $y = x+5$ are continuous, so again only possible discontinuity is where they join.

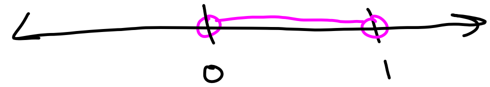
$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x+5) = 1+5 = 6$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (5x) = 5(1) = 5$

Because $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$, the $\lim\limits_{x\to 1} f(x)$ does not exist.

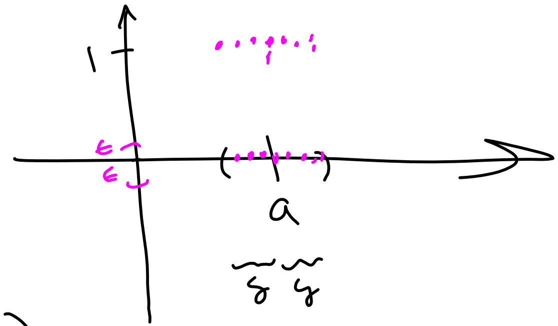
Because $5 = f(1) = \lim\limits_{x\to 1^+} f(x)$, f is continuous from the right.

$\boxed{\text{Jump discontinuity at } x = 1}$



Example 17: Determine the values of x, if any, at which the function is discontinuous.

$$g(x)=\begin{cases}1 & x \text{ rational}\\ 0 & x \text{ irrational}\end{cases}$$



Discontinuous everywhere!

(oscillating Discontinuity)

Example 18: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\begin{cases}\dfrac{|x|}{x} & x \neq 0\\ 3 & x = 0\end{cases}$$

Example 19: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & x \neq 5\\ 10 & x = 0\end{cases}$$

Example 20: Find a function g that agrees with f for $x \neq 1$ and is continuous on $(-\infty, \infty)$.

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Domain of f: $x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

$$f(x) = \frac{(x+1)(x-1)}{(x-1)}$$

"cancelled" version of this function is $\boxed{g(x) = x+1.}$

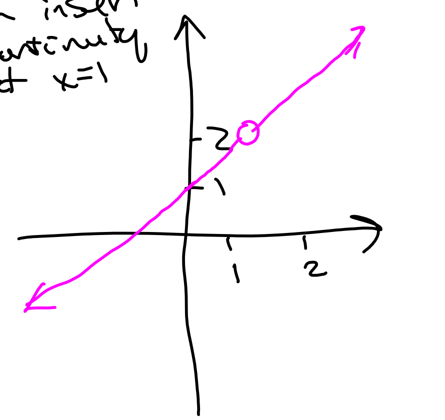
Domain of g: $(-\infty, \infty)$

g agrees with f except at $x=1$

Let's graph f:

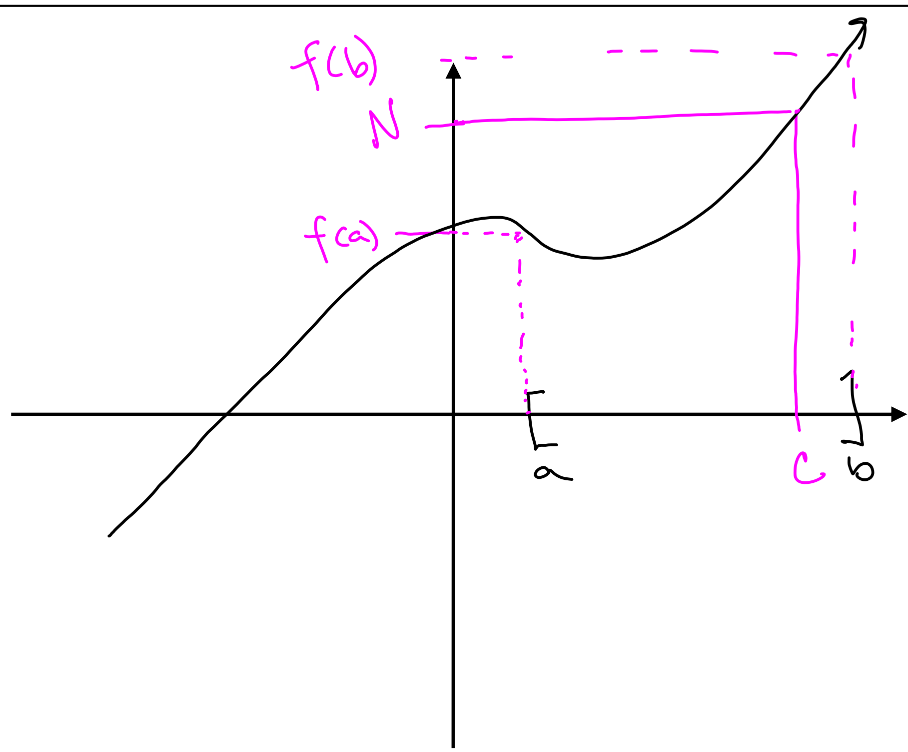
graph $g(x) = x+1$ 1st

Then insert a removable discontinuity at $x=1$



The Intermediate Value Theorem:

Suppose that *f* is continuous on the closed interval $[a, b]$ and let *N* be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number *c* in (a, b) such that $f(c) = N$.



Example 21: Show that $f(x) = x^3 - 4x^2 + 6$ has a zero between 1 and 2.

$f(1) = 1^3 - 4(1)^2 + 6 = 3$

$f(2) = 2^3 - 4(2)^2 + 6 = 8 - 16 + 6 = -2$

f is continuous on $(-\infty, \infty)$, so certainly continuous on $[1, 2]$.

0 is between $f(1) = 3$ and $f(2) = -2$

So, from Intermediate Value Theorem (IVT), there must be a c in $[1, 2]$ such that $f(c) = 0$.

Example 22: Is there a number that is equal to its own cosine?

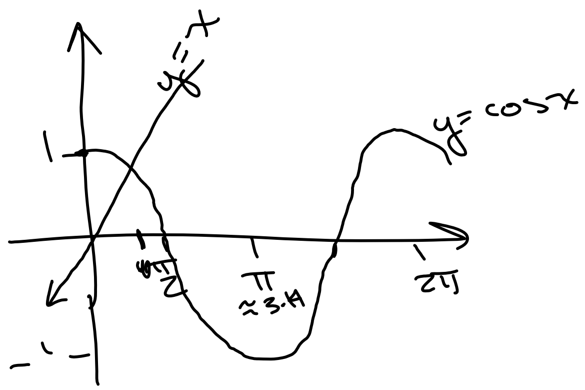
In other words, does $x = \cos x$ have a solution? Let $f(x) = x - \cos x$. Then, does $f(x)$ have a zero?

$f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ (pos)

$f(0) = 0 - \cos 0 = 0 - 1 = -1$ (neg)

So, from Int. Value Thm, there must be a c in the interval $[0, \frac{\pi}{2}]$ such that $f(c) = 0$.

Yes, there is a solution.



Homework Qs

1.3 #85 Find $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

for $f(x) = x^2 - 4x$. This is called a difference quotient.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\overbrace{(x+\Delta x)^2 - 4(x+\Delta x)}^{f(x+\Delta x)} - \overbrace{(x^2-4x)}^{f(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{4x} - 4\Delta x - \cancel{x^2} + \cancel{4x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x - 4)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x + 0 - 4 = \boxed{2x - 4}$$