1.5: Infinite Limits

There are two types of limits involving infinity. $\lim_{x\to\infty} f(x)$ $\lim_{x\to -\infty} f(x)$

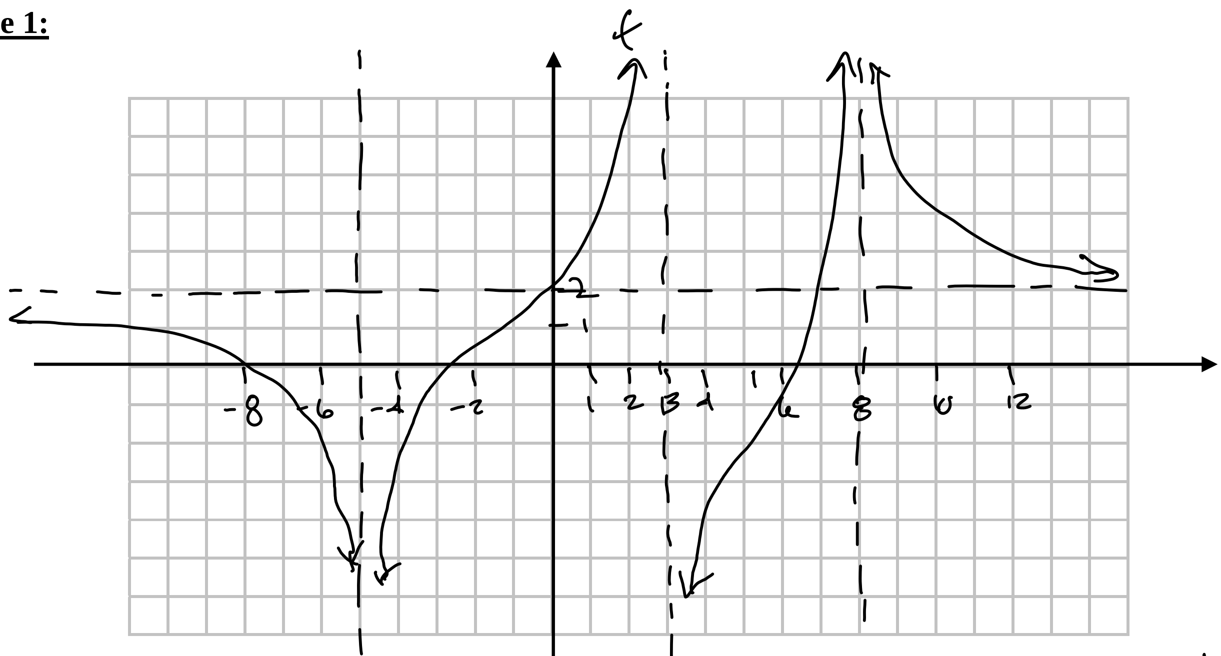
Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes and will be covered in Section 3.5, as we learn to graph functions.

Infinite limits take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits can result in vertical asymptotes, also important in graphing functions.

Can also have $\lim_{x\to\pm\infty} f(x) = \pm\infty$ (describes end behavior of a graph)

Determining infinite limits from a graph:

Example 1:



$\lim_{x\to -5^-} f(x) = -\infty$

$\lim_{x\to -5^+} f(x) = -\infty$

so $\lim_{x\to -5} f(x) = -\infty$

none of these exist

$\lim_{x\to 3^-} f(x) = \infty$

$\lim_{x\to 3^+} f(x) = -\infty$

limits don't exist

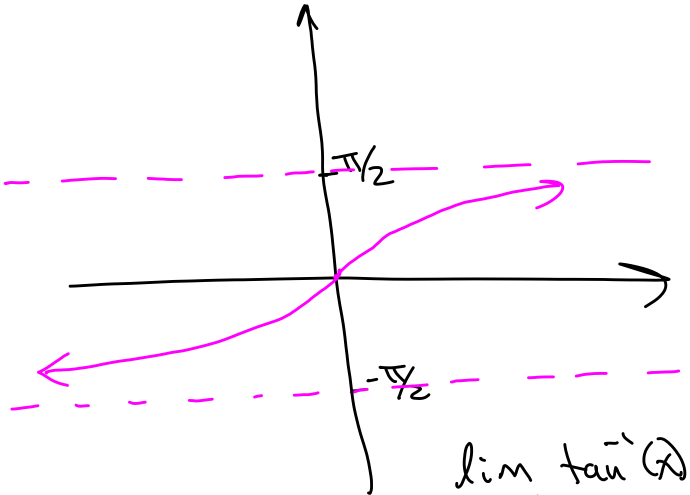
$\lim_{x\to 3} f(x)$ does not exist

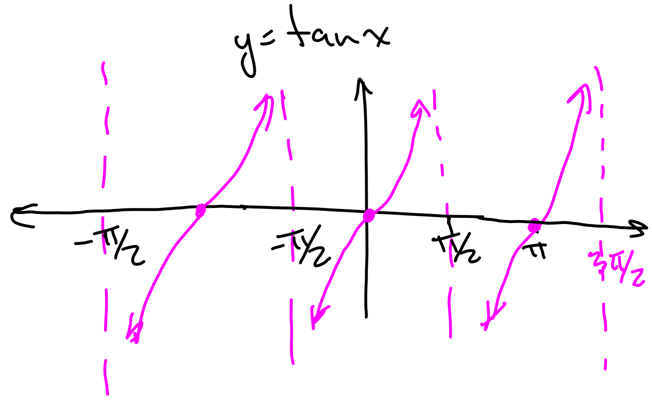
$\lim_{x\to 8} f(x) = \infty$ (doesn't exist)

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to -\infty} f(x) = 2$

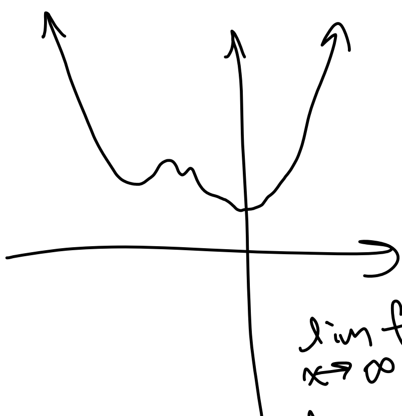
Ex: $f(x) = \tan^{-1}x = \arctan(x)$





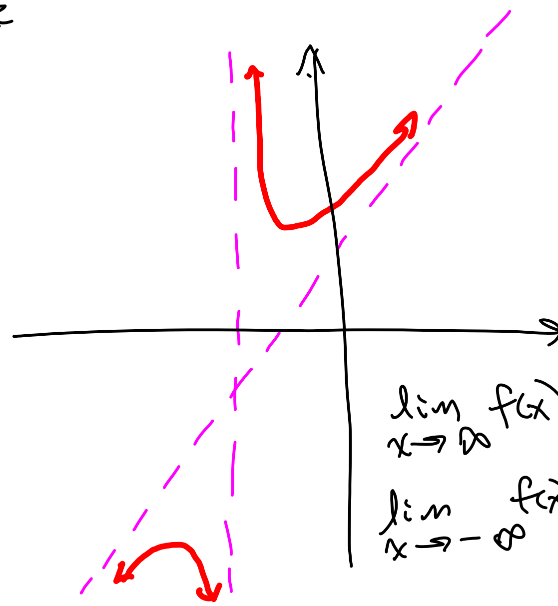
$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$



$$\lim_{x\to\infty} f(x) = +\infty$$

$$\lim_{x\to-\infty} f(x) = +\infty$$



$$\lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

Determining infinite limits from a table of values:

Example 2: Use a table of values to determine $\cancel{\lim_{x\to 2} f(x) = \frac{x+6}{x-2}}$.

Direct Sub: $\frac{8}{0}$ (this is not indeterminate)

$\lim_{x\to 2} \frac{x+6}{x-2}$

x	$\frac{x+6}{x-2}$
1.5	−15
1.9	−79
1.95	−159
1.99	−799
1.999	−7999 etc
2.5	
2.1	81
2.05	161
2.01	801
2.001	8001
2.0001	80001

From table

$\lim_{x\to 2^-} \frac{x+6}{x-2} = -\infty$, $\lim_{x\to 2^+} \frac{x+6}{x-2} = +\infty$ (limits don't exist)

So, $\lim_{x\to 2} \frac{x+6}{x-2}$ does not exist

Calculate without table

As $x \to 2^-$, $y \to \frac{1.99+6}{1.99-2} \to \frac{8}{-\text{tiny}} \to -\text{huge}$

So $\boxed{\lim_{x\to 2^-} \frac{x+6}{x-2} = -\infty}$

As $x \to 2^+$, $y \to \frac{2.01+6}{2.01-2} \to \frac{8}{+\text{tiny}} \to +\text{huge}$

So $\boxed{\lim_{x\to 2^+} \frac{x+6}{x-2} = +\infty}$

Example 3: Use ~~a table~~ of values logic to determine $\lim_{x\to 2} \left(\cancel{f(x)} = \frac{x+6}{(x-2)^2}\right)$

Find $\lim_{x\to 2} \frac{x+6}{(x-2)^2}$

As $x \to 2^-$, $y \to \frac{1.99+6}{(1.99-2)^2} \to \frac{8}{(-\text{tiny})^2} \to \frac{8}{+\text{tiny}} \to +\text{huge}$

So $\boxed{\lim_{x\to 2^-} \frac{x+6}{(x-2)^2} = +\infty}$

As $x \to 2^+$, $y \to \frac{2.01+6}{(2.01-2)^2} \to \frac{8}{(+\text{tiny})^2} \to \frac{8}{+\text{tiny}} \to +\text{huge}$

So $\boxed{\lim_{x\to 2^+} \frac{x+6}{(x-2)^2} = +\infty}$

(none of these limits exist)

Thus, $\boxed{\lim_{x\to 2} \frac{x+6}{(x-2)^2} = +\infty}$

Important:

Statements such as $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, or $\lim_{x\to a^+} f(x) = -\infty$ do NOT mean the limit exists. Rather, these statements mean that the limit DOES NOT EXIST. and they describe the reason that the limit fails to exist (by describing the behavior of the function near the given x-value).

Formal definition of an infinite limit:

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a} f(x) = \infty$$

means that for every positive real number *M*, there exists a number $\delta > 0$ such that

$$f(x) > M \text{ whenever } 0 < |x-a| < \delta.$$

Similarly, $\lim_{x\to a} f(x) = \cancel{\infty}\ -\infty$ means that for every negative real number *N*, there exists a number $\delta > 0$ such that

$$f(x) < N \text{ whenever } 0 < |x-a| < \delta.$$

Example 4: Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Scratchwork: Let $M > 0$. Determine the δ.

want ~~Suppose~~ $\frac{1}{x^2} > M$ (we want $f(x) > M$

$1 > Mx^2$ Note $\sqrt{x^2} = |x|$

$\frac{1}{M} > x^2$

$\sqrt{\frac{1}{M}} > |x|$

$|x| < \frac{1}{\sqrt{M}}$

$|x-0| < \frac{1}{\sqrt{M}}$. Let $\delta = \frac{1}{\sqrt{M}}$

Proof: Let $M > 0$. Let $\delta = \frac{1}{\sqrt{M}}$.

Suppose $0 < |x-0| < \delta$

$|x| < \delta$. Then $-\delta < x < \delta$

$-\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}}$

Because $x < \frac{1}{\sqrt{M}}$, then $x\sqrt{M} < 1$

$x^2 M < 1$

$M < \frac{1}{x^2}$ $\boxtimes$

Evaluating infinite limits from an equation:

Direct Sub: $\frac{4-8}{4-4} \to \frac{-4}{0}$

Limit does not exist

(but need to look at each side separately to see if you get ∞ or $-\infty$)

Example 5: Determine $\lim\limits_{x\to 4}\dfrac{x-8}{x-4}$.

As $x \to 4^-$, $y \to \dfrac{3.99-8}{3.99-4} \to \dfrac{-4}{-\text{tiny}} \to +\text{huge}$

So $\boxed{\lim\limits_{x\to 4^-}\dfrac{x-8}{x-4} = \infty}$

As $x \to 4^+$, $y \to \dfrac{4.01-8}{4.01-4} \to \dfrac{-4}{+\text{tiny}} \to -\text{huge}$

So $\boxed{\lim\limits_{x\to 4^+}\dfrac{x-8}{x-4} = -\infty}$

So $\boxed{\lim\limits_{x\to 4}\dfrac{x-8}{x-4} \text{ does not exist}}$

(sides don't match)

Example 6: Determine $\lim\limits_{x\to 2^-}\dfrac{x-8}{x^2-4}$.

As $x \to 2^-$, $y \to \dfrac{1.99-8}{(1.99)^2-4} \to \dfrac{-6}{-\text{tiny}} \to +\infty$

$\boxed{\lim\limits_{x\to 2^-}\dfrac{x-8}{x^2-4} = +\infty}$ (does not exist)

Example 7: Determine $\lim\limits_{x\to 3}\dfrac{x^3-2}{(x-3)^2}$.

As $x \to 3^-$, $y \to \dfrac{(2.99)^3-2}{(2.99-3)^2} \to \dfrac{27-2}{(-\text{tiny})^2} \to \dfrac{25}{+\text{tiny}} \to +\text{huge}$. So $\lim\limits_{x\to 3^-}\dfrac{x^3-2}{(x-3)^2} = +\infty$

As $x \to 3^+$, $y \to \dfrac{(3.01)^3-2}{(3.01-3)^2} \to \dfrac{27-2}{(+\text{tiny})^2} \to \dfrac{25}{+\text{tiny}} \to +\text{huge}$. So $\lim\limits_{x\to 3^+}\dfrac{x^3-2}{(x-3)^2} = +\infty$

$\therefore \boxed{\lim\limits_{x\to 3}\dfrac{x^3-2}{(x-3)^2} = \infty}$

Example 8: Determine $\lim\limits_{x\to \pi^+}\dfrac{\sin\left(\frac{x}{3}\right)}{1+\cos x}$.

$\lim\limits_{x\to \pi^+}\dfrac{\sin(x/3)}{1+\cos x}$

As $x \to \pi^+$, $y \to \dfrac{\sin \pi/3}{1-0.99} \to \dfrac{\sqrt{3}/2}{1-0.99} \to \dfrac{\sqrt{3}/2}{+\text{tiny}} \to +\infty$

As $x \to \pi^-$, $y \to \dfrac{\sin \pi/3}{1-0.99} \to \dfrac{\sqrt{3}/2}{+\text{tiny}} \to +\text{huge}$ $+\infty$

So $\boxed{\lim\limits_{x\to \pi}\dfrac{\sin(x/3)}{1+\cos x} = +\infty}$

Note

$\theta > \frac{\pi}{6} \Rightarrow 3\theta > \frac{3\pi}{6} \Rightarrow 3\theta > \frac{\pi}{2}$

$\Rightarrow$ Quadrant II $\Rightarrow \cos 3\theta$ is negative

Example 9: Determine $\lim\limits_{x\to \pi/6}\tan(3x)$.

$\lim\limits_{x\to \frac{\pi}{6}}\tan(3x) = \lim\limits_{x\to \pi/6}\dfrac{\sin(3x)}{\cos(3x)}$.

As $x \to \frac{\pi}{6}^+$, $y \to \dfrac{\sin\frac{3\pi}{6}}{-\text{tiny}} \to \dfrac{1}{-\text{tiny}} \to -\text{huge}$

So $\lim\limits_{x\to \pi/6^+}\tan(3x) = -\infty$

As $x \to \frac{\pi}{6}^-$, $y \to \dfrac{\sin(3\pi/6)}{+\text{tiny}} \to \dfrac{1}{+\text{tiny}} \to +\text{huge}$. So $\lim\limits_{x\to \frac{\pi}{6}^-}\tan(3x) = \infty$

see next page

Ex 9 cont'd:

Therefore $\lim_{x \to \pi/6} \tan(3x)$ does not exist. You could also think about graph of $y = \tan(3x)$ to see limit does not exist.

Vertical asymptotes:

<u>Vertical Asymptotes</u>:

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 10: Determine the asymptotes of $f(x) = \dfrac{x-2}{x+3}$. Sketch the graph.

Vertical asymptote: $x = -3$

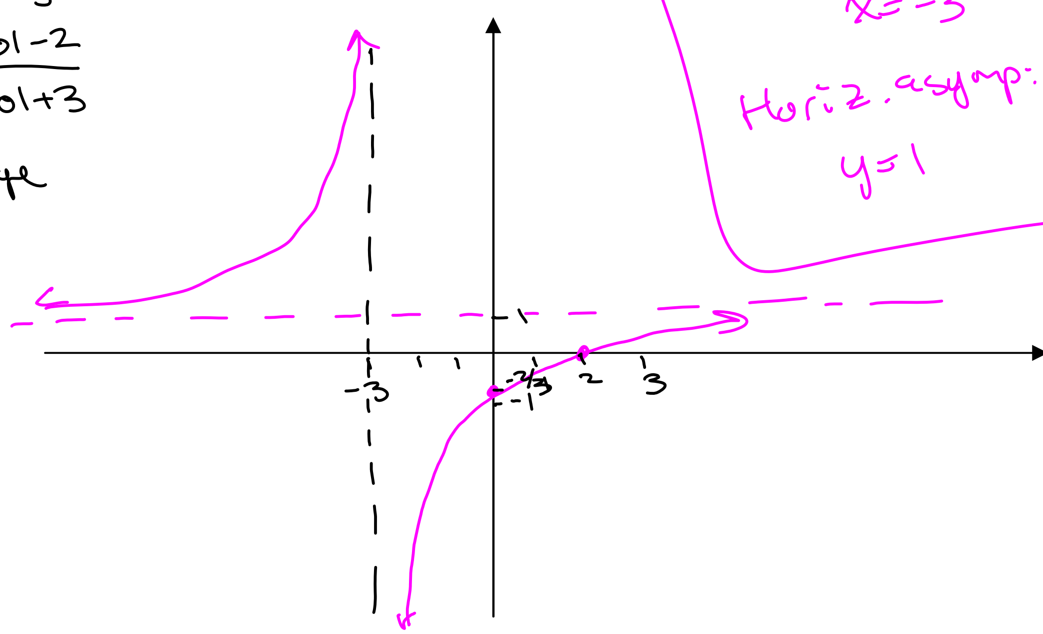
Horiz. asymp: $y = 1$

Undefined at $x = -3$

as $x \to -3^-$, $y \to \dfrac{-3.01-2}{-3.01+3}$

$\to \dfrac{-5}{-\text{tiny}} \to +\text{huge}$

so $\lim_{x \to -3^-} f(x) = +\infty$



As $x \to -3^+$,

$y \to \dfrac{-2.99-2}{-2.99+3}$

$\to \dfrac{-5}{+\text{tiny}}$

$\to -\text{huge}$

So $\lim_{x \to -3^+} f(x) = -\infty$

From Algebra / Precalc the horizontal asymptote is $y = 1$.

$f(x) = \dfrac{x-2}{x+3}$

As $x \to \pm\infty$, $y \to \dfrac{x}{x} = 1$

So $y = 1$ is the horizontal asymptote.

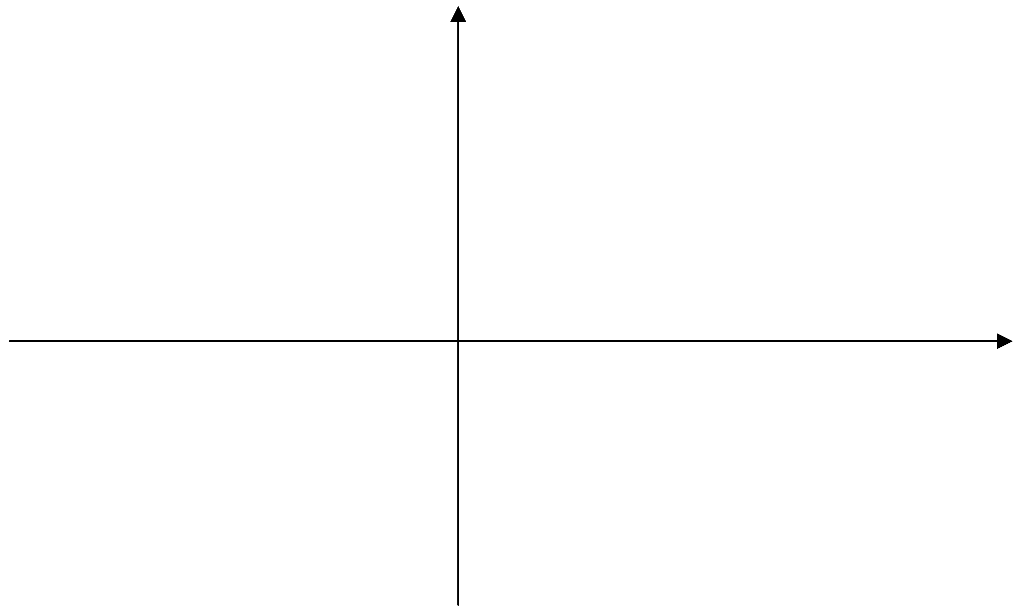
Set $x = 0$ to find y-intercept:

$y = \dfrac{0-2}{0+3} = -\dfrac{2}{3}$

Set $y = 0$ to find x-intercept: $0 = \dfrac{x-2}{x+3} \Rightarrow 0 = x-2 \Rightarrow x = 2$

x-intercept: 2

Example 11: Determine the vertical asymptotes of $f(x)=\dfrac{3}{x^2-1}$. Sketch the graph.



Example 12: Determine the vertical asymptotes of $f(x)=\dfrac{x^2-9}{x^2-5x+6}$. Sketch the graph.