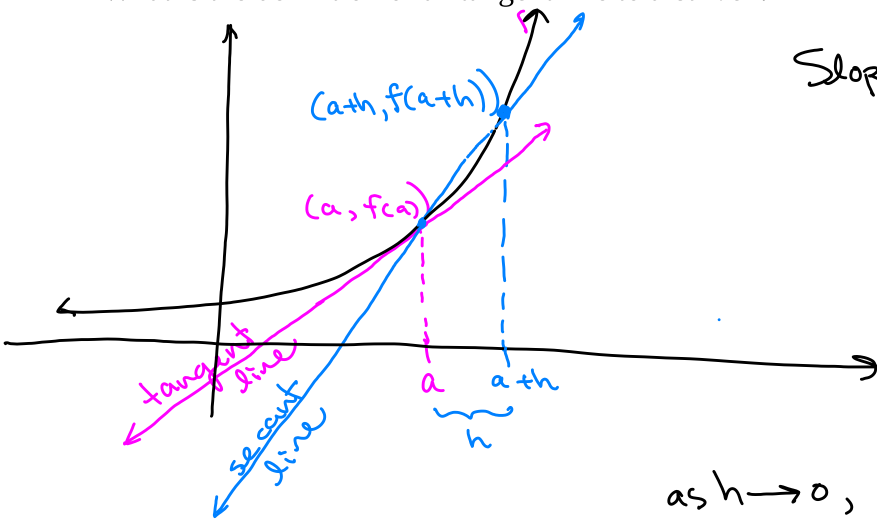
2.1: The Derivative and the Tangent Line Problem

What is the definition of a "tangent line to a curve"?



Slope of secant line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{a+h-a}$$

$$\boxed{\text{slope of secant line} = \frac{f(a+h) - f(a)}{h}}$$

as $h \to 0$, the slope of the secant line approaches the slope of the tangent line.

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

Definition: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{provided this limit exists.}$$

Equivalently,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{provided this limit exists.}$$

Note: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x = a$.

Note: The slope of the tangent line to the graph of f at the point $(a, f(a))$ is also called the slope of the graph of f at $x = a$.

How to get the second expression for slope: Instead of using the points $(a, f(a))$ and $(x, f(x))$ on the secant line and letting $x \to a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \to 0$.

Check: $y\Big|_{x=3} = 4(3)^2+1 = 37$ ✓

Example 1: Find the slope of the curve $y = 4x^2 + 1$ at the point $(3, 37)$. Find the equation of the tangent line at this point.

$f(x) = 4x^2+1$

1st point: $(3, 37)$
2nd point: $(3+h, f(3+h))$

Find slope:
$$m = \lim_{h\to 0} \frac{f(3+h)-f(3)}{3+h-3} = \lim_{h\to 0} \frac{\overbrace{4(3+h)^2+1}^{f(3+h)} - \overbrace{(4(3)^2+1)}^{f(3)}}{h}$$

$$= \lim_{h\to 0} \frac{4(9+6h+h^2)+1-37}{h} = \lim_{h\to 0} \frac{\cancel{36}+24h+4h^2-\cancel{36}}{h}$$

$$= \lim_{h\to 0} \frac{24h+4h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(24+4h)}{\cancel{h}} = \lim_{h\to 0}(24+4h) = 24+4(0) = 24+0 = 24$$

Find eqn of tangent line with slope $m = 24$ and $(x_1, y_1) = (3, 37)$ $\Rightarrow$
$y - y_1 = m(x - x_1)$
$y - 37 = 24(x-3)$
$y - 37 = 24x - 72$

$$\boxed{y = 24x - 35}$$

Example 2: Find an equation of the tangent line to the curve $y = x^3$ at the point $(1,1)$.

Alternative Definition
$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ = slope of tangent line

Here, $a = 1$.

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x^3-1^3}{x-1}$$

$$= \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+1x+1^2)}{\cancel{x-1}} = \lim_{x\to 1}(x^2+x+1) = 1^2+1+1 = 3$$

Recall: Factorization of sum/difference of 2 cubes:
$a^3+b^3 = (a+b)(a^2-ab+b^2)$
$a^3-b^3 = (a-b)(a^2+ab+b^2)$

Find eqn: Using $y = mx+b$ with $m = 3$, $x = 1$, $y = 1$:
$1 = 3(1)+b$
$-2 = b$. so eqn is $\boxed{y = 3x-2}$

Example 3: Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at the point where $x = 2$.

The derivative:

The derivative of a function at x is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at x.

"f - prime"

Definition: The *derivative* of a function f at x is the function f' whose value at x is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$ provided this limit exists.

The process of finding derivatives is called differentiation. To differentiate a function means to find its derivative.

Equivalent ways of defining the derivative:

equivalent to $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ (Our book uses this one. It is identical to the definition above, except uses Δx in place of h.)

Δx = "change in x"

Δ = "delta" = "change in"

$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x}$$

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$ (Gives the derivative at the specific point where $x = a$.)

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ (Gives the derivative at the specific point where $x = a$.)

$g(x) = \frac{x^2}{3} - \frac{6x}{3} = \frac{1}{3}x^2 - 2x$

Example 4: Suppose that $g(x) = \dfrac{x^2-6x}{3}$. Determine $g'(x)$ and $g'(3)$.

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\overbrace{\tfrac{1}{3}(x+h)^2-2(x+h)}^{g(x+h)} - \overbrace{\left[\tfrac{1}{3}x^2-2x\right]}^{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\tfrac{1}{3}(x^2+2xh+h^2)-2x-2h-\tfrac{1}{3}x^2+2x}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{\tfrac{1}{3}x^2}+\tfrac{2}{3}xh+\tfrac{1}{3}h^2-\cancel{2x}-2h-\cancel{\tfrac{1}{3}x^2}+\cancel{2x}}{h} = \lim_{h\to 0}\frac{\tfrac{2}{3}xh+\tfrac{1}{3}h^2-2h}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}\left(\tfrac{2}{3}x+\tfrac{1}{3}h-2\right)}{\cancel{h}} = \lim_{h\to 0}\left(\tfrac{2}{3}x+\tfrac{1}{3}h-2\right) = \tfrac{2}{3}x+\tfrac{1}{3}(0)-2 = \tfrac{2}{3}x-2$$

$$\boxed{g'(x) = \tfrac{2}{3}x-2}$$

$$g'(3) = \tfrac{2}{3}(3)-2 = 2-2 = 0$$

$$\boxed{g'(3) = 0}$$

Example 5: Suppose that $f(x)=\sqrt{x^2+1}$. Find the equation of the tangent line at the point where $x=2$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

$$=\lim_{h\to 0}\frac{\overbrace{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}^{a-b}}{h}\cdot\left(\frac{\overbrace{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}^{a+b}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right)=\lim_{h\to 0}\frac{\overbrace{(x+h)^2+1}^{a^2}-\overbrace{(x^2+1)}^{b^2}}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{1}-\cancel{x^2}-\cancel{1}}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}=\lim_{h\to 0}\frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}=\lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

$$=\frac{2x+0}{\sqrt{(x+0)^2+1}+\sqrt{x^2+1}}=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}=f'(x)$$

At the point where $x=2$, slope $=m=f'(2)=\dfrac{2}{\sqrt{2^2+1}}=\dfrac{2}{\sqrt5}=\dfrac{2\sqrt5}{5}$.

Find y-value: $f(2)=\sqrt{2^2+1}=\sqrt5$

Find eqn of tangent line:

$$y-y_1=m(x-x_1)$$

$$y-\sqrt5=\frac{2\sqrt5}{5}(x-2)$$

$$y-\sqrt5=\frac{2\sqrt5}{5}x-\frac{4\sqrt5}{5}$$

$$y=\frac{2\sqrt5}{5}x-\frac{4\sqrt5}{5}+\sqrt5\left(\frac{5}{5}\right)$$

$$y=\frac{2\sqrt5}{5}x-\frac{4\sqrt5}{5}+\frac{5\sqrt5}{5}$$

$$\boxed{y=\frac{2\sqrt5}{5}x+\frac{\sqrt5}{5}}$$

Example 6: Determine the equation of the tangent line to $f(x)=\dfrac{x-2}{x^2+1}$ at the point $\left(-2,-\dfrac{4}{5}\right)$.

Summary:

The slope of the secant line between two points is often called a <u>difference quotient</u>. The <u>difference quotient of *f* at *a*</u> can be written in either of the forms below.

$$\frac{f(x)-f(a)}{x-a} \qquad\qquad \frac{f(a+h)-f(a)}{h}.$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of *f* between the two points.

The <u>derivative of *f* at *a*</u> is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.

2) the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$.

3) the (instantaneous) rate of change of f with respect to *x* at *a*.

4) $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)

5) $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)

Common notations for the derivative of $y = f(x)$:

Calc III

$$f'(x) \qquad \frac{d}{dx}f(x) \qquad y' \qquad D_x f(x) \qquad \boxed{\frac{dy}{dx}} \qquad Df(x)$$

The notation $\dfrac{dy}{dx}$ was created by Gottfried Wilhelm Leibniz and means $\dfrac{dy}{dx} = \lim\limits_{\Delta x\to 0}\cancel{\dfrac{\Delta x}{\Delta y}}$. $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$

To evaluate the derivative at a particular number *a*, we write

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiability:

Definition: A function *f* is *differentiable* at *a* if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

> Theorem: If *f* is differentiable at *a*, then *f* is continuous at *a*.

Note: The converse is not true—there are functions that are continuous at a number but not differentiable.

Note: Open intervals: (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$.

Closed intervals: $[a,b]$, $(-\infty,a]$, $[a,\infty)$, $(-\infty,\infty)$.

To discuss differentiability on a closed interval, we need the concept of a *one-sided derivative*.

Derivative from the left: $\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$ $\quad \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$

Derivative from the right: $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ $\quad \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$

For a function f to be differentiable on the closed interval $[a,b]$, it must be differentiable on the open interval (a,b). In addition, the derivative from the right at *a* must exist, and the derivative from the left at *b* must exist.

Ways in which a function can fail to be differentiable:

1. Sharp corner
2. Cusp
3. Vertical tangent
4. Discontinuity

Cusp: There is a cusp at $x=a$ if

$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=+\infty$ and $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=-\infty$ ($f(a)$ is defined)

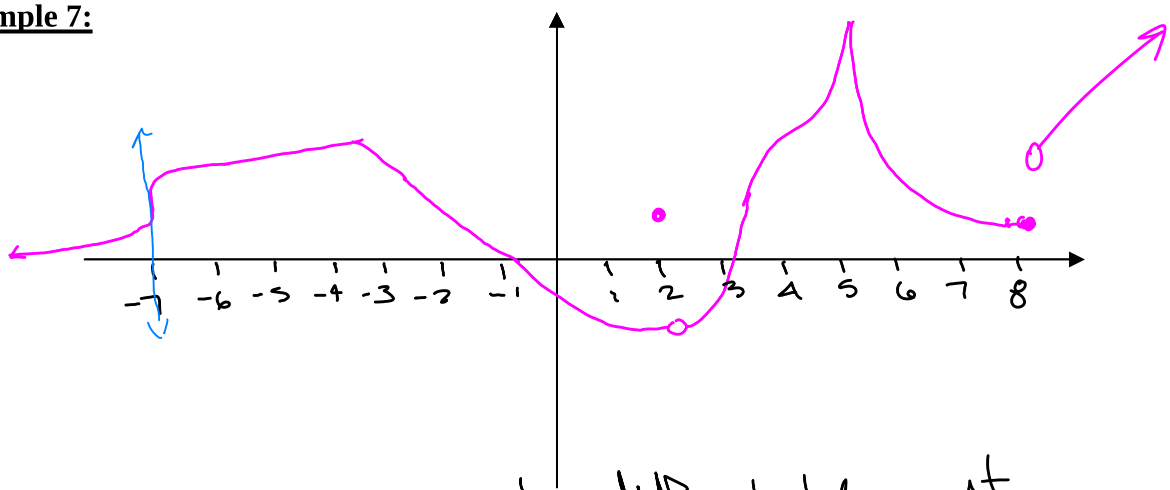
but f is continuous at $x=a$

OR

$\lim_{x\to a^-}\frac{f(x)-f(a)}{x}=-\infty$ and $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=+\infty$

(Slopes of tangent lines approach vertical)

Example 7:

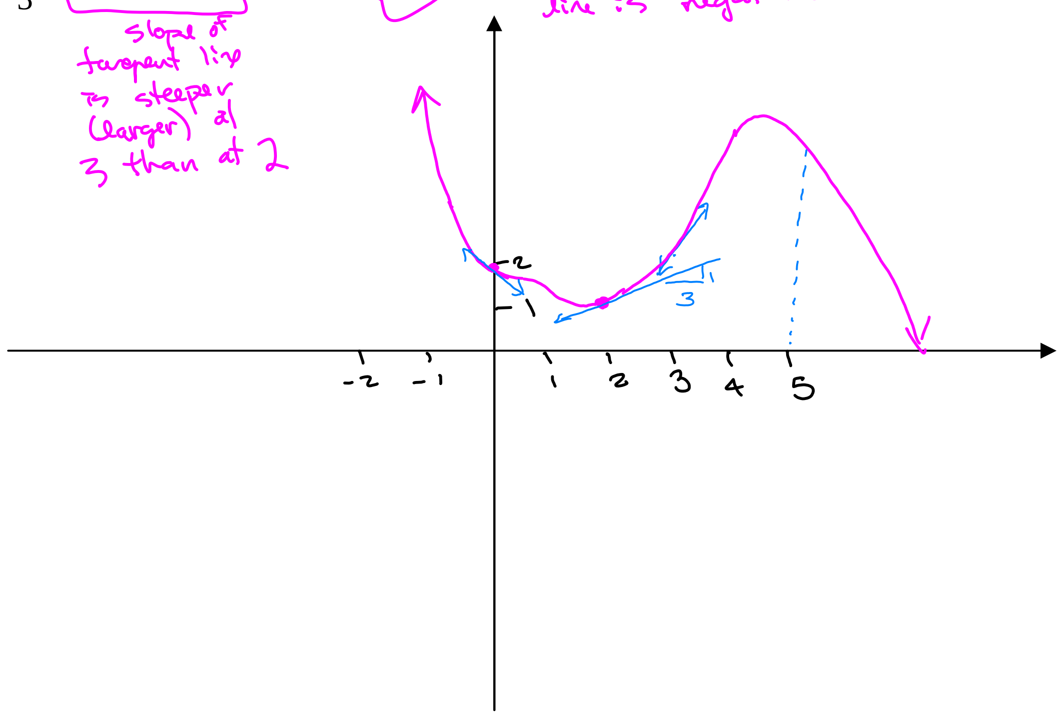


This function is not differentiable at

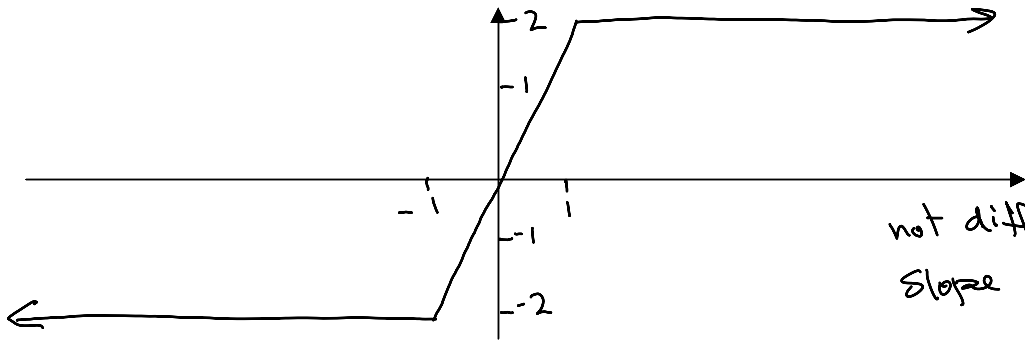
$x = -3.5$ (sharp corner), $x = -7$ (vertical tangent), $x = 2$ (discontinuity), $x = 5$ (cusp)

and $x = 8$ (discontinuity)

Example 8: Sketch the graph of a function for which $f(0) = 2$, $f'(0) = -1$ → slope of tangent line is −1, $f(2) = 1$, $f'(2) = \frac{1}{3}$, $\underbrace{f'(3) > f'(2)}_{\text{slope of tangent line is steeper (larger) at 3 than at 2}}$, and $f'(5) < 0$ → slope of tangent line is negative



Example 9: Use the graph of the function to draw the graph of the derivative.

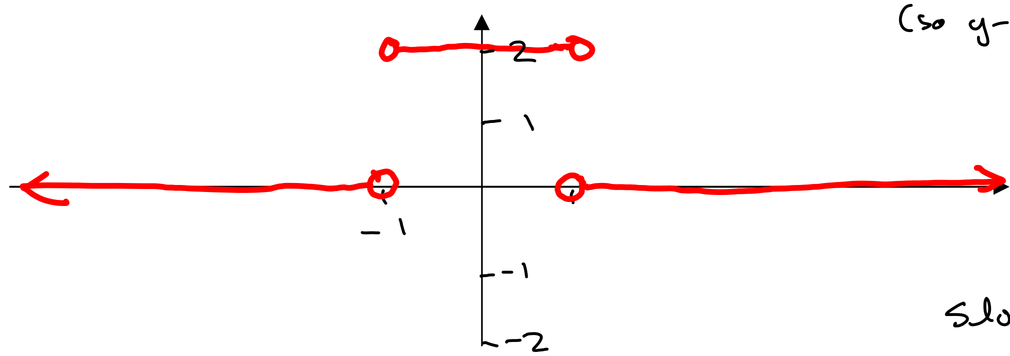


f

not differentiable at ± 1

Slope is 0 for $(1, \infty), (-\infty, -1)$

(so y-values for f' are 0)

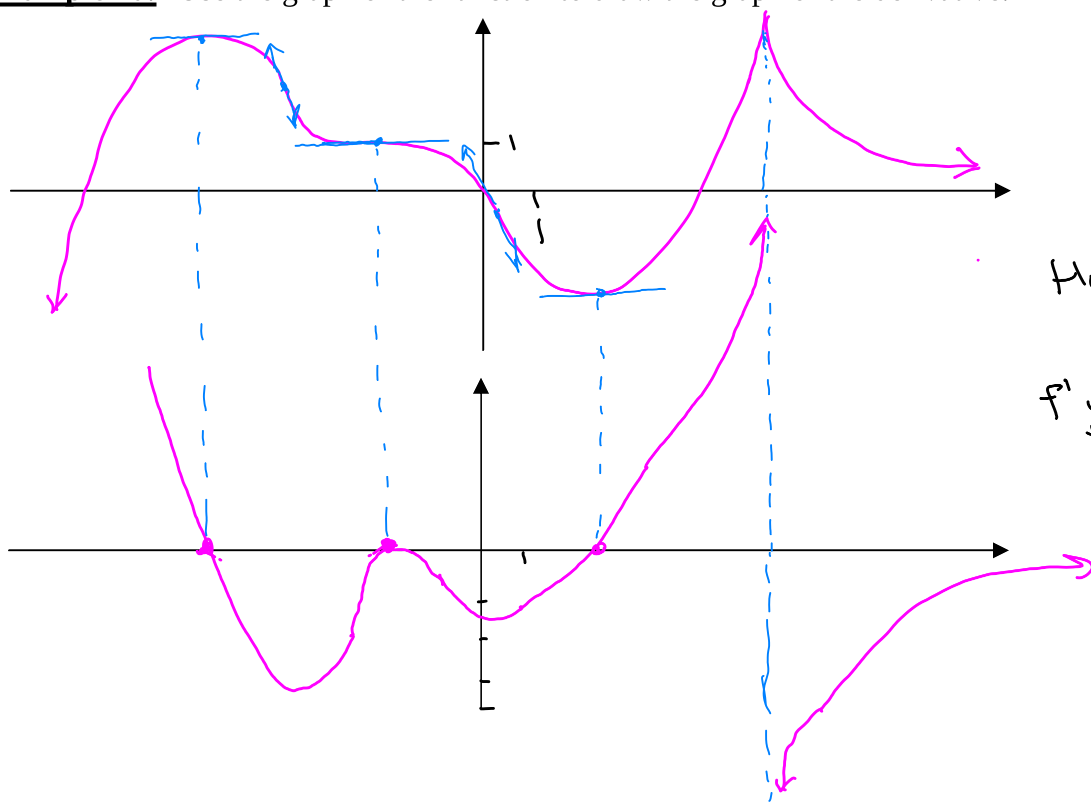


f'

Slope is +2 for interval $(-1, 1)$

(so the outputs (y-values) for f' are 2 on $(-1,1)$)

Example 10: Use the graph of the function to draw the graph of the derivative.



f

Horizontal tangent line $\Rightarrow f'(x) = 0$

f' undefined at the corner/cusp

f'