2.2: Basic Differentiation Rules and Rates of Change

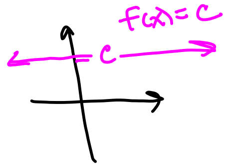
Definition of derivative:
(most common def'n)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

Basic differentiation formulas:

1. $\frac{d}{dx}(c) = 0$ for any constant c.



2. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n.
3. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
4. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
5. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 1: Find the derivative of $f(x) = 7$.

$$f'(x) = 0$$

Ex $1\frac{1}{2}$:

$$f(x) = x^4$$
$$f'(x) = 4x^{4-1}$$
$$= \boxed{4x^3}$$
(Rule #2)

Ex $1\frac{2}{3}$:

$$g(x) = 7x^3$$
$$g'(x) = 7\frac{d}{dx}(x^3)$$
$$= 7(3x^2)$$
$$= \boxed{21x^2}$$

Example 2: Find the derivative of $f(x) = 5x^3 - x^7 + 12x$.

$$f'(x) = 15x^2 - 7x^6 + 12x^{1-1}$$
$$= 15x^2 - 7x^6 + 12x^0$$
$$= 15x^2 - 7x^6 + 12(1) = \boxed{15x^2 - 7x^6 + 12}$$

Example 3: Find the derivative of $g(x) = x^{17} + x^{3/2}$.

$$g(x) = x^{17} + x^{3/2}$$
$$g'(x) = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-1}$$
$$= \boxed{17x^{16} + \frac{3}{2}x^{1/2}} = \boxed{17x^{16} + \frac{3}{2}\sqrt{x}}$$

<u>Recall</u>:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$
$$\frac{1}{x^n} = x^{-n}$$

<u>Example 4:</u> Find the derivative of $f(x) = \sqrt[5]{x} + \frac{1}{x^2}$.

Rewrite as powers: $f(x) = x^{1/5} + x^{-2}$

$f'(x) = \frac{1}{5}x^{\frac{1}{5}-1} - 2x^{-2-1} = \frac{1}{5}x^{-4/5} - 2x^{-3}$

$= \boxed{\frac{1}{5\sqrt[5]{x^4}} - \frac{2}{x^3}}$

<u>Example 5:</u> Find the derivative of $f(x) = \frac{2}{\sqrt[4]{x}}$.

$f(x) = \frac{2}{\sqrt[4]{x}}$

Rewrite: $f(x) = 2x^{-1/4}$ $f'(x) = 2\left(-\frac{1}{4}\right)x^{-\frac{1}{4}-1} = -\frac{2}{4}x^{-5/4}$

$= -\frac{1}{2x^{5/4}} = -\frac{1}{2(x^5)^{1/4}}$

$= \boxed{-\frac{1}{2\sqrt[4]{x^5}}}$

<u>Example 6:</u> Find the derivative of $h(x) = \left(\sqrt{x}\right)^5$.

$h(x) = (\sqrt{x})^5 = \left(x^{1/2}\right)^5 = x^{5/2}$

$h'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \boxed{\frac{5}{2}x^{\frac{3}{2}}}$

$= \boxed{\frac{5\sqrt{x^3}}{2}}$

<u>Example 7:</u> Find the derivative of $f(x) = -\sqrt[3]{6x^4}$.

$f(x) = -\sqrt[3]{6x^4} = -\sqrt[3]{6}\sqrt[3]{x^4}$

$= -6^{1/3}x^{4/3}$

$f'(x) = -6^{1/3} \cdot \frac{4}{3}x^{\frac{4}{3}-1} = -6^{1/3} \cdot \frac{4}{3}x^{1/3} = -\frac{4(6x)^{1/3}}{3}$

$= \boxed{-\frac{4\sqrt[3]{6x}}{3}}$

<u>Example 8:</u> Find the derivative of $f(x) = \frac{10}{x^4}$.

$f(x) = 10x^{-4}$

$f'(x) = -40x^{-4-1} = -40x^{-5}$

$= \boxed{-\frac{40}{x^5}}$

Example 9: Find the derivative of $g(x)=\dfrac{2\sqrt{x}}{7}$.

$g(x) = \frac{2}{7}x^{1/2}$

$g'(x) = \frac{2}{7}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) = \frac{2}{14}x^{-1/2} = \frac{1}{7}x^{-1/2}$

$= \boxed{\dfrac{1}{7\sqrt{x}}}$

Example 10: Find the derivative of $f(t)=\dfrac{3}{4t^2}-\sqrt[3]{7t}$.

$f(t) = \frac{3}{4}t^{-2} - 7^{1/3}t^{1/3}$

$f'(t) = \frac{3}{4}(-2t^{-3}) - 7^{1/3}\cdot\frac{1}{3}t^{\frac{1}{3}-1} = -\frac{6}{4}t^{-3} - 7^{1/3}\cdot\frac{1}{3}t^{-2/3}$

$= \boxed{-\dfrac{3}{2t^3} - \dfrac{\sqrt[3]{7}}{3\sqrt[3]{t^2}}}$

Example 11: Find the derivative of $f(u)=\dfrac{7u^5+u^2-9\sqrt{u}}{u^2}$.

$f(u) = \frac{7u^5}{u^2} + \frac{u^2}{u^2} - \frac{9u^{1/2}}{u^2} = 7u^3 + 1 - 9u^{\frac{1}{2}-2}$

$= 7u^3 + 1 - 9u^{-3/2}$

$f'(u) = 21u^2 + 0 - 9\left(-\frac{3}{2}\right)u^{-3/2-1}$

$= \boxed{21u^2 + \frac{27}{2}u^{-5/2}} = 21u^2 + \dfrac{27}{2\sqrt{u^5}}$

Example 12: Find the equation of the tangent line to the graph of $f(x)=3x-x^2$ at the point $(-2,-10)$.

$f'(x) = 3 - 2x$

Find slope: $m = f'(-2) = 3-2(-2) = 3+4 = 7$

$y - y_1 = m(x - x_1)$

$y - (-10) = 7(x-(-2))$

$y + 10 = 7(x+2)$

$y + 10 = 7x + 14$

$\boxed{y = 7x + 4}$

Example 13: Find the point(s) on the graph of $f(x) = x^2 + 6x$ where the tangent line is horizontal.

Tangent line horizontal $\Rightarrow f'(x) = 0$

$$f'(x) = 2x + 6$$

Set $f'(x) = 0$: $2x + 6 = 0$

$2x = -6$

$x = -3$

Find the y-value: $f(-3) = (-3)^2 + 6(-3) = 9 - 18 = -9$

$\boxed{\text{Tangent line is horizontal at } (-3, -9).}$

Definition: The *normal line* to a curve at the point P is defined to be the line passing through P that is perpendicular to the tangent line at that point.

Example 14: Determine the equation of the normal line to the curve $y = \dfrac{1}{x}$ at the point $\left(3, \dfrac{1}{3}\right)$.

Recall: The slopes of perpendicular lines are opposite reciprocals.

So, we first find slope of tangent line:

$$y = \frac{1}{x} = x^{-1}$$

Slope of tangent line is: $\dfrac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\dfrac{1}{x^2}$

Slope of tangent line at $(3, \frac{1}{3})$ is $-\dfrac{1}{3^2} = -\dfrac{1}{9} = m_1$

Slope of normal line is $m_2 = +\dfrac{9}{1} = 9$

Find eqn: $y - y_1 = m(x - x_1) \Rightarrow y - \frac{1}{3} = 9(x - 3)$

$y - \frac{1}{3} = 9x - 27$

$y = 9x - 27 + \frac{1}{3}$

$y = 9x - \frac{81}{3} + \frac{1}{3}$

$\boxed{y = 9x - \frac{80}{3}}$

Derivatives of trigonometric functions:

$\dfrac{d}{dx}(\sin x) = \cos x$	$\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
$\dfrac{d}{dx}(\cos x) = -\sin x$	$\dfrac{d}{dx}(\sec x) = \sec x \tan x$
$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\dfrac{d}{dx}(\cot x) = -\csc^2 x$

☆ Memorize these!

Note: Derivatives of all the co-functions have a minus sign.

Example 15: Find the derivative of $y = 2\cos x - 4\tan x$.

$$\frac{dy}{dx} = 2(-\sin x) - 4\sec^2 x = \boxed{-2\sin x - 4\sec^2 x}$$

Example 16: Find the derivative of $y = \dfrac{\sin x}{4} + 3x^4 + \pi^2$.

$$y = \frac{1}{4}\sin x + 3x^4 + \pi^2$$

$$\frac{dy}{dx} = \frac{1}{4}\cos x + 12x^3 + 0 = \boxed{\frac{1}{4}\cos x + 12x^3}$$

Example 17: Determine the equation of the tangent line to the graph of $y = \sec x$ at the point where $x = \dfrac{\pi}{4}$.

Find the derivative: $\dfrac{dy}{dx} = \sec x \tan x$

Slope at $x = \frac{\pi}{4}$ is: $m = \left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{4}}$

$$= \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} \cdot \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} \cdot \frac{\sqrt{2}/2}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} \cdot (1) = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Find y-value: $\left. y\right|_{x=\frac{\pi}{4}} = \sec\frac{\pi}{4} = \sqrt{2}$

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$\boxed{y = \sqrt{2}\,x - \frac{\pi\sqrt{2}}{4} + \sqrt{2}}$$

Example 18: Find the points on the curve $y = \tan x - 2x$ where the tangent line is horizontal.

Set $\dfrac{dy}{dx} = 0$: (slope of a horizontal line is 0).

$$\frac{dy}{dx} = \sec^2 x - 2$$

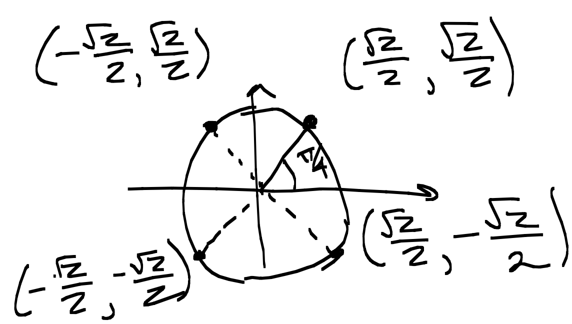
$$\sec^2 x - 2 = 0$$

$$(\sec(x))^2 = 2$$

$$\sec(x) = \pm\sqrt{2}$$

$$\cos(x) = \pm\frac{1}{\sqrt{2}}$$

$$\cos(x) = \pm\frac{\sqrt{2}}{2}$$



On $[0, 2\pi]$, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\to$ on $(-\infty, \infty)$

$$\boxed{x = \frac{(2k+1)\pi}{4}, \text{ where } k \text{ is any integer}}$$

(odd multiples of $\frac{\pi}{4}$)

The derivative as a rate of change:

The <u>average rate of change</u> of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h = x_1 - x_0 \neq 0.$$

This is the same as the slope of the secant line joining points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$.

The <u>instantaneous rate of change</u> (or, equivalently, just the <u>rate of change</u>) of f when $x = a$ is the slope of the tangent line to graph of f at the point $(a, f(a))$.

Therefore, the instantaneous rate of change is given by the <u>derivative</u> f'.

Volume of sphere $= \frac{4}{3}\pi r^3$

<u>Example 19:</u> Find the average rate of change in volume of a sphere with respect to its radius r (r = radius) as r changes from 3 to 4. Find the instantaneous rate of change when the radius is 3.

Average rate of change $= \dfrac{\text{change in Volume}}{\text{change in radius}} = \dfrac{V_2 - V_1}{r_2 - r_1} = \dfrac{\frac{256\pi}{3} - 36\pi\left(\frac{3}{3}\right)}{4-3}$

$r_1 = 3 \Rightarrow V_1 = \frac{4}{3}\pi(3)^3 = \frac{4\pi(9)}{1} = 36\pi$

$r_2 = 4 \Rightarrow V_2 = \frac{4}{3}(\pi)(4)^3 = \frac{256\pi}{3}$

$= \dfrac{\frac{256\pi}{3} - \frac{108\pi}{3}}{1}$

$= \boxed{\frac{148\pi}{3}}$ avg rate of change.

Instantaneous rate of change:

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = V' = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)$

$= 4\pi r^2$

$\left.\frac{dV}{dr}\right|_{r=3} = V'(3) = 4\pi(3)^2$

$= \boxed{36\pi}$

<u>Example 20:</u> Find the rate of change of the area of a circle with respect to (a) the diameter; (b) the circumference.

(b) on next page

$A = \text{area} = \pi r^2$, $r = \text{radius}$

a) write A in terms of d = diameter

$d = 2r$

$\frac{1}{2}d = r$

substitute into $A = \pi r^2$:

$A = \pi\left(\frac{1}{2}d\right)^2$

$A = \frac{\pi}{4}d^2$

Instantaneous Rate of change $= \frac{dA}{d(d)} = A'(d) = \frac{\pi}{4}(2d) = \boxed{\frac{\pi d}{2}}$

Ex 20 b: $A = \pi r^2$

C = circumference = $2\pi r$

solve $C = 2\pi r$ for r:

$\frac{C}{2\pi} = r$

Substitute into $A = \pi r^2$:

$A = \pi \left(\frac{C}{2\pi}\right)^2 = \frac{\pi C^2}{4\pi^2} = \frac{C^2}{4\pi} = \frac{1}{4\pi} C^2$

Inst. Rate of change:

$\frac{dA}{dC} = A'(C) = \frac{1}{4\pi}(2C) = \boxed{\frac{C}{2\pi}}$

Velocity:

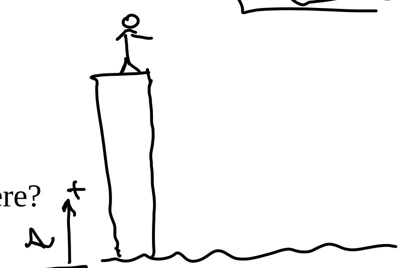
If the independent variable represents *time*, then the derivative can be used to analyze motion.

If the function $s(t)$ represents the position of an object, then the derivative $s'(t) = \frac{ds}{dt}$ is the velocity of the object.

(The velocity is the instantaneous rate of change in distance. The average velocity is the average rate of change in distance.)

Example 21: A person stands on a bridge 40 feet above a river. He throws a ball vertically upward with an initial velocity of 50 ft/sec. Its height (in feet) above the river after t seconds is $s = -16t^2 + 50t + 40$.

a) What is the velocity after 3 seconds?
b) How high will it go?
c) How long will it take to reach a velocity of 20 ft/sec?
d) When will it hit the water? How fast will it be going when it gets there?



s = height above river

$s(t) = -16t^2 + 50t + 40$

velocity: $v(t) = \frac{ds}{dt} = s'(t) = -32t + 50$

a) After 3 seconds:

$v(3) = -32(3) + 50 = -96 + 50 = \boxed{-46 \text{ ft/sec}}$

(it's going down)

b) at maximum height: velocity is 0.

set $v'(t) = 0$: $0 = -32t + 50$

$32t = 50$

$t = \frac{50}{32} = \frac{25}{16} = 1\frac{9}{16}$ sec

Find position when $t = \frac{25}{16}$:

$s(t) = -16t^2 + 50t + 40$

$s\left(\frac{25}{16}\right) = -16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) + 40 = \boxed{79.0625 \text{ ft}}$

c) when does it reach velocity of 20 ft/sec?

Set $v(t) = s'(t) = 20$:

$-32t + 50 = 20$

$-32t = -30$

$t = \frac{-30}{-32} = \boxed{\frac{15}{16}}$ sec

d) When does it hit the water?

Set position $=0$: $-16t^2+50t+40=0$

$$t=\frac{-50\pm\sqrt{50^2-4(-16)(40)}}{2(-16)} \Rightarrow t=3.785 \text{ sec}$$
$$t\approx -0.66 \text{ sec}$$

Only $t=3.785$ sec makes sense in our problem.

It takes 3.785 sec to hit water.
It is going 71.12 ft/sec ft/sec when it hits.

$v(t)=-32t+50$

$v(3.785)=-32(3.785)+50$

$=-71.12$ ft/sec

Note:
speed $=|\text{velocity}|$

Example 22: Suppose a bullet is shot straight up at an initial velocity of 73 feet per second. If air resistance is neglected, its height from the ground (in feet) after t seconds is given by $h(t) = -16.1t^2 + 73t$.

a. The velocity after 2 seconds.
b. How high will the bullet go?
c. When will the bullet reach the ground?
d. How fast will it be traveling when it hits the ground?

Example 23: Suppose the position of a particle is given by $f(t) = t^4 - 32t + 7$. What is the velocity after 3 seconds? When is the particle at rest?