

2.3: Product and Quotient Rules and Higher Order Derivatives

The Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

“the first times the derivative of the second plus the second times the derivative of the first”

Example 1: Find the derivative of $f(x) = x^7(4x^3)$.

Product Rule

$$\begin{aligned} f'(x) &= x^7 \frac{d}{dx}(4x^3) + (4x^3) \frac{d}{dx}(x^7) \\ &= x^7(12x^2) + 4x^3(7x^6) \\ &= 12x^9 + 28x^9 = \boxed{40x^9} \end{aligned}$$

Simplifying 1^{st} :

$$f(x) = 4x^{10}$$

$$f'(x) = \boxed{40x^9}$$

Example 2: Find the derivative of $f(x) = (4x^3 + x^2 - 2)(x^4 + 8)$.

$$\begin{aligned} f'(x) &= (4x^3 + x^2 - 2) \frac{d}{dx}(x^4 + 8) + (x^4 + 8) \frac{d}{dx}(4x^3 + x^2 - 2) \\ &= (4x^3 + x^2 - 2)(4x^3 + 0) + (x^4 + 8)(12x^2 + 2x + 0) \end{aligned}$$

Example 3: Find the derivative of $f(x) = \sqrt{x}(x^5 - 3x^2 + 12x)$.

$$f(x) = x^{1/2}(x^5 - 3x^2 + 12x)$$

$$\begin{aligned} f'(x) &= x^{1/2} \frac{d}{dx}(x^5 - 3x^2 + 12x) + (x^5 - 3x^2 + 12x) \frac{d}{dx}(x^{1/2}) \\ &= x^{1/2}(5x^4 - 6x + 12) + (x^5 - 3x^2 + 12x) \left(\frac{1}{2} x^{-1/2}\right) \end{aligned}$$

Then distribute & clean it up.

Example 4: Find the derivative of $(4x^3 + 1)(\sqrt{x} + \frac{1}{x} - 2x)$.

Using product rule,

$$\begin{aligned} \frac{dy}{dx} &= (4x^3 + 1) \frac{d}{dx} \left(x^{1/2} + x^{-1} - 2x\right) + \left(x^{1/2} + \frac{1}{x} - 2x\right) \frac{d}{dx}(4x^3 + 1) \\ &= (4x^3 + 1) \left(\frac{1}{2}x^{-1/2} - x^{-2} - 2\right) + \left(x^{1/2} + \frac{1}{x} - 2x\right)(12x^2) \end{aligned}$$

(It is more efficient to distribute the $x^{1/2}$ at the beginning, and then you wouldn't need Product Rule)

Again, this one is easier to do by distributing, instead of using product rule.

The Quotient Rule:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Using Properties of Exponents and The Power Rule:

Example 5: Find the derivative of $f(x) = \frac{-x^5}{4x^3}$.

Using Quotient Rule:

$$f'(x) = \frac{4x^3 \frac{d}{dx}(-x^5) - (-x^5) \frac{d}{dx}(4x^3)}{(4x^3)^2}$$

$$= \frac{4x^3(-5x^4) + x^5(12x^2)}{16x^6} = \frac{-20x^7 + 12x^7}{16x^6} = \frac{-8x^7}{16x^6}$$

$$f(x) = \frac{-x^5}{4x^3} = -\frac{x^2}{4} = -\frac{1}{4}x^2$$

$$f'(x) = -\frac{1}{4}(2x) = \boxed{-\frac{1}{2}x}$$

same!

Example 6: Find the derivative of $g(x) = \frac{5-3x}{6x+7}$.

$$g'(x) = \frac{(6x+7) \frac{d}{dx}(5-3x) - (5-3x) \frac{d}{dx}(6x+7)}{(6x+7)^2}$$

$$= \frac{(6x+7)(-3) - (5-3x)(6)}{(6x+7)^2} = \frac{-18x - 21 - 6(5-3x)}{(6x+7)^2}$$

$$= \frac{-18x - 21 - 30 + 18x}{(6x+7)^2} = \boxed{\frac{-51}{(6x+7)^2}}$$

Example 7: Find the derivative of $f(x) = \frac{\sqrt{x}}{x^3 - x^4}$.

$$f(x) = \frac{x^{1/2}}{x^3 - x^4}$$

$$f'(x) = \frac{(x^3 - x^4) \frac{d}{dx}(x^{1/2}) - x^{1/2} \frac{d}{dx}(x^3 - x^4)}{(x^3 - x^4)^2} = \frac{(x^3 - x^4)(\frac{1}{2}x^{-1/2}) - x^{1/2}(3x^2 - 4x^3)}{(x^3 - x^4)^2}$$

$$= \frac{\frac{1}{2}x^{5/2} - \frac{1}{2}x^{7/2} - 3x^{5/2} + 4x^{7/2}}{(x^3 - x^4)^2} = \frac{-\frac{5}{2}x^{5/2} + \frac{7}{2}x^{7/2}}{(x^3 - x^4)^2} \cdot \frac{2}{2}$$

$$= \frac{-5x^{5/2} + 7x^{7/2}}{2(x^3 - x^4)^2} = \frac{x^{5/2}(-5 + 7x)}{2(x^3(1-x))^2} = \frac{x^{5/2}(-5 + 7x)}{2x^6(1-x)^2} = \boxed{\frac{-5 + 7x}{2x^{7/2}(1-x)^2}}$$

Example 8: Find the derivative of $f(x) = x^2 \sin x$.

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx} (\sin x) + (\sin x) \frac{d}{dx} (x^2) \\ &= x^2 \cos x + (\sin x)(2x) \\ &= \boxed{x^2 \cos x + 2x \sin x} \end{aligned}$$

Example 9: Find the derivative of $f(x) = x + \sin x \cos x$.

$$f'(x) = 1 + \sin x \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (\sin x)$$

Example 10: Find the derivative of $y = \frac{1 - \cos x}{\sin x}$.

$$\begin{aligned} y' = \frac{dy}{dx} &= \frac{\sin x \frac{d}{dx} (1 - \cos x) - (1 - \cos x) \frac{d}{dx} (\sin x)}{(\sin x)^2} \\ &= \frac{\sin x (0 - (-\sin x)) - (1 - \cos x)(\cos x)}{\sin^2 x} \\ &= \frac{\sin x (\sin x) - \cos x + \cos^2 x}{\sin^2 x} = \frac{\overbrace{\sin^2 x + \cos^2 x}^1 - \cos x}{\sin^2 x} \end{aligned}$$

Example 11: Prove that $\frac{d}{dx} (\csc x) = -\csc x \cot x$.

$$\begin{aligned} &= \frac{1 - \cos x}{\sin^2 x} = \frac{1 - \cos x}{1 - \cos^2 x} \\ &= \frac{1 - \cancel{\cos x}}{(1 + \cancel{\cos x})(1 - \cancel{\cos x})} \\ &= \boxed{\frac{1}{1 + \cos x}} \end{aligned}$$

Higher order derivatives:

Once the derivative of $f(x)$ is also a function, it is possible to find the derivative of $f'(x)$ too. This is called the *second derivative* and is denoted $f''(x)$. The second derivative gives the instantaneous rate of change of the derivative. In other words, it tells us how fast the slope is changing.

Similarly, the derivative of $f''(x)$ can be calculated and this is called the *third derivative* $f'''(x)$.

In general, we can keep calculating the derivative of the previous derivative. The *n th derivative* is found by taking the derivative n times. The *n th derivative of f* is denoted $f^{(n)}(x)$.

Other notation: $y', y'', y''', \dots, y^{(n)}$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$$

$$D'y, D^2y, D^3y, \dots, D^ny$$

$f^{(4)}(x)$ is the 4th derivative

Example 12: Suppose that $y = -7x^5 + 6x^4 - \frac{2}{x}$. Find y''' .

$$y = -7x^5 + 6x^4 - 2x^{-1}$$

$$y' = -35x^4 + 24x^3 + 2x^{-2}$$

$$y'' = -140x^3 + 72x^2 - 4x^{-3}$$

$$y''' = -420x^2 + 144x + 12x^{-4}$$

$$y''' = -420x^2 + 144x + \frac{12}{x^4}$$

Example 13: Suppose that $f(x) = \sqrt[3]{x}$. Find the second and third derivatives.

$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f''(x) = \frac{1}{3} \left(-\frac{2}{3}x^{-5/3} \right) = -\frac{2}{9}x^{-5/3}$$

$$f'''(x) = -\frac{2}{9} \left(-\frac{5}{3}x^{-8/3} \right) = \frac{10}{27}x^{-8/3}$$

$$f''(x) = -\frac{2}{9\sqrt[3]{x^5}} = -\frac{2}{9x^{5/3}}$$

$$f'''(x) = \frac{10}{27\sqrt[3]{x^8}} = \frac{10}{27x^{8/3}}$$

Using derivatives to describe the motion of an object:

If the dependent variable t represents time, and the function $s(t)$ represents the position (distance from a particular point) of an object, then

- the velocity $v(t)$ is the first derivative $s'(t) = \frac{ds}{dt}$.
- the acceleration $a(t)$ is the second derivative $s''(t) = \frac{dv}{dt}$.
- the jerk $j(t)$ is the third derivative $s'''(t) = \frac{da}{dt}$.
- the speed is the absolute value of the velocity $|v(t)| = \left| \frac{ds}{dt} \right|$.

Example 14: The position (in feet) of an object is given by $s(t) = t^4 - 32t + 7$, with t measured in seconds. Find functions representing the velocity, acceleration, and jerk. Find the velocity and acceleration after 1, 2, and 3 seconds.

$$\text{position: } s(t) = t^4 - 32t + 7$$

$$\text{velocity: } v(t) = \frac{ds}{dt} = 4t^3 - 32$$

$$\text{acceleration: } a(t) = \frac{dv}{dt} = s''(t) = 12t^2$$

$$\text{jerk: } j(t) = \frac{da}{dt} = s'''(t) = 24t.$$

$$v(1) = 4(1)^3 - 32 = -28 \text{ ft/sec}$$

$$a(1) = 12(1)^2 = 12 \text{ ft/sec}^2$$

$$v(2) = 4(2)^3 - 32 = 0 \text{ ft/sec}$$

$$a(2) = 12(2)^2 = 48 \text{ ft/sec}^2$$

$$v(3) = 4(3)^3 - 32 = 108 - 32 = 76 \text{ ft/sec}$$

$$a(3) = 12(3)^2 = 108 \text{ ft/sec}^2$$