2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Example 1: Suppose $g(x)=(x^3+2)^2$. Find $g'(x)$.

$g(x) = x^6 + 4x^3 + 4$

$g'(x) = 6x^5 + 12x^2$

What if you had: $h(x) = (x^3+2)^{50}$?

The chain rule lets us differentiate composite functions.

> The Chain Rule:
>
> If *f* and *g* are both differentiable and $F(x) = f(g(x))$, then *F* is differentiable and
>
> $$F'(x) = f'\big(g(x)\big)\,g'(x).$$
>
> Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

With the chain rule, we take the derivative of the "outer function" and multiply by the derivative of the "inner function".

Example 2: Find the derivative of $h(x)=(x^3+2)^{50}$.

$h(x) = (x^3+2)^{50}$

$h'(x) = 50(x^3+2)^{49}\frac{d}{dx}(x^3+2)$

$= 50(x^3+2)^{49}(3x^2)$

$= \boxed{150x^2(x^3+2)^{49}}$

Note: $y = x^{50}$

$\frac{dy}{dx} = 50x^{49}$

Note: Let $u = x^3+2$

For $y = (x^3+2)^{50}$, then $y = u^{50}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= 50u^{49}\cdot 3x^2$

$= 50(x^3+2)^{49}\cdot 3x^2$

Example 3: Find the derivative of $f(x)=\sqrt{2x^3-5x}$.

$f(x)=(2x^3-5x)^{1/2}$

$f'(x)=\frac{1}{2}(2x^3-5x)^{-1/2}\cdot\frac{d}{dx}(2x^3-5x)$

$=\frac{1}{2}(2x^3-5x)^{-1/2}(6x^2-5) = \boxed{\dfrac{6x^2-5}{2\sqrt{2x^3-5x}}}$

Example 4: Suppose that $y=\dfrac{2}{(3x^5-4)^3}$. Find $\dfrac{dy}{dx}$.

$y=\dfrac{2}{(3x^5-4)^3} = 2(3x^5-4)^{-3}$

$\frac{dy}{dx}=-6(3x^5-4)^{-4}(15x^4)$

$=-90x^4(3x^5-4)^{-4} = \boxed{-\dfrac{90x^4}{(3x^5-4)^4}}$

Ex $4\frac{1}{2}$ $y=\dfrac{7}{6x+8}$

Rewrite:

$y=7(6x+8)^{-1}$

$\frac{dy}{dx}=-7(6x+8)^{-2}(6)$

$=-42(6x+8)^{-2}$

$=\boxed{\dfrac{-42}{(6x+8)^2}}$

Example 5: Find the derivative of $f(x)=\sqrt[3]{\cos x}$.

$f(x)=(\cos x)^{1/3}$

$f'(x)=\frac{1}{3}(\cos x)^{-2/3}\frac{d}{dx}(\cos x)$

$=\frac{1}{3}(\cos x)^{-2/3}(-\sin x) = -\frac{1}{3}\sin x(\cos x)^{-2/3}$

$=\boxed{-\dfrac{\sin x}{3\sqrt[3]{\cos^2 x}}}$

Example 6: Find the derivative of $g(x)=\cos\sqrt[3]{x}$.

$g(x)=\cos(\sqrt[3]{x}) = \cos(x^{1/3})$

$g'(x)=-\sin(x^{1/3})\frac{d}{dx}(x^{1/3}) = -(\sin(x^{1/3}))(\frac{1}{3}x^{-2/3})$

Note: For $y=\cos x$

$\frac{dy}{dx}=-\sin x$

$=-\frac{1}{3}x^{-2/3}\sin(x^{1/3})$

$=\boxed{-\dfrac{\sin(\sqrt[3]{x})}{3\sqrt[3]{x^2}}}$

Example 7: Suppose that $y=\dfrac{1}{\cos x}$. Find $\dfrac{dy}{dx}$.

$y=(\cos x)^{-1}$

$\frac{dy}{dx}=-1(\cos x)^{-2}\frac{d}{dx}(\cos x)$

$=-1(\cos x)^{-2}(-\sin x)$

$=\dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \boxed{\sec x\tan x}$

Ex.: $g(x) = \sin(6x)$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

$g'(x) = \cos(6x)\frac{d}{dx}(6x)$

$= (\cos(6x))(6) = \boxed{6\cos(6x)}$

Example 8: Suppose that $h(x) = (3x^2-4)^3(2x-9)^2$. Find $h'(x)$.

$h'(x) = (3x^2-4)^3\frac{d}{dx}(2x-9)^2 + (2x-9)^2\frac{d}{dx}(3x^2-4)^3$

$= (3x^2-4)^3(2)(2x-9)\frac{d}{dx}(2x-9) + (2x-9)^2(3)(3x^2-4)^2\frac{d}{dx}(3x^2-4)$

$= (3x^2-4)^3(2)(2x-9)(2) + (2x-9)^2(3)(3x^2-4)^2(6x)$

$= 4(3x^2-4)^3(2x-9) + 18x(2x-9)^2(3x^2-4)^2$

$= 2(2x-9)(3x^2-4)^2[2(3x^2-4) + 9x(2x-9)]$

$= 2(2x-9)(3x^2-4)^2[6x^2-8+18x^2-81x]$

$= \boxed{2(2x-9)(3x^2-4)^2(24x^2-81x-8)}$

Example 9: Suppose that $f(x) = \left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

Example 10: Find the derivative of $y = \cos(\sin(\pi x^2))$.

$\frac{dy}{dx} = -\sin(\sin(\pi x^2))\frac{d}{dx}(\sin(\pi x^2))$

$= -\sin(\sin(\pi x^2))(\cos(\pi x^2))\frac{d}{dx}(\pi x^2)$

$= -\sin(\sin(\pi x^2))(\cos(\pi x^2))(2\pi x)$

$= \boxed{-2\pi x(\sin(\sin(\pi x^2)))(\cos(\pi x^2))}$

Note: $y = \cos(x)$

$\frac{dy}{dx} = -\sin(x)\cdot\frac{d}{dx}(x) = -\sin(x)(1) = -\sin x$

Example 11: Find the derivative of $g(x) = \dfrac{\cos^2 x}{\sqrt{4x+1}}$.

Example 12: Find the first and second derivatives of $f(x) = (x^2 + 4)^5$.

Example 13: Find the first and second derivatives of $f(x) = \cos\left(3x^2\right)$.

Example 14: Find the first and second derivatives of $y = \dfrac{2}{(3x+5)^2}$

Example 15: Find the equation of the tangent line to $y = \dfrac{x^2}{\sin \pi x + 1}$ at the point where $x = 1$.