2.5: Implicit Differentiation

Example 1: Given the equation $x^3 - 4y - 9x^2 = 5$, find $\frac{dy}{dx}$ by

a) Solving explicitly for y.
b) Implicit differentiation.

a) solve for y:

$$-4y = 5 - x^3 + 9x^2$$

$$y = -\frac{5}{4} + \frac{1}{4}x^3 - \frac{9}{4}x^2$$

$$\frac{dy}{dx} = 0 + \frac{1}{4}(3x^2) - \frac{9}{4}(2x) = \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

b) $x^3 - 4y - 9x^2 = 5$

$$\frac{d}{dx}(x^3 - 4y - 9x^2) = \frac{d}{dx}(5)$$

$$3x^2 - 4 \cdot \frac{d}{dx}(y) - 18x = 0$$

$$3x^2 - 4\frac{dy}{dx} - 18x = 0$$

solve for $\frac{dy}{dx}$:

$$-4\frac{dy}{dx} = -3x^2 + 18x$$

Divide by -4:

$$\frac{dy}{dx} = \frac{-3x^2}{-4} + \frac{18x}{-4}$$

$$= \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

Example 2: Find $\frac{dy}{dx}$ for $xy = 4$.

Implicit diff:

$$xy = 4$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

Product rule:

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Example 3: Find $\frac{dy}{dx}$ for the equation $x^3y - 2x^2y^3 + x^2 - 3 = 0$.

$$\frac{d}{dx}\left(\underbrace{x^3}\underbrace{y} - 2\underbrace{x^2}_{1^{st}}\underbrace{y^3}_{2^{nd}} + x^2 - 3\right) = \frac{d}{dx}(0)$$

Product rule:

$$x^3\frac{d}{dx}(y) + y\frac{d}{dx}(x^3) - 2\left[x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2)\right] + 2x - 0 = 0$$

$$x^3\frac{dy}{dx} + y(3x^2) - 2\left[x^2 3y^2\frac{dy}{dx} + y^3(2x)\right] + 2x = 0$$

Ex: $4x^2 - 6x^3 + 5y^3 = 18x - 5$

Implicit differentiation:

Note: $y = \frac{1}{5}\sqrt[3]{18x - 5 - 4x^2 + 6x^3}$

Note: $\frac{d}{dx}(y) = \frac{dy}{dx}$

$$\frac{d}{dx}(4x^2 - 6x^3 + 5y^3) = \frac{d}{dx}(18x - 5)$$

$$8x - 18x^2 + 15y^2\frac{dy}{dx} = 18$$

solve for $\frac{dy}{dx}$:

$$15y^2\frac{dy}{dx} = 18 - 8x + 18x^2$$

$$\boxed{\frac{dy}{dx} = \frac{18 - 8x + 18x^2}{15y^2}}$$

Ex: $3x^2 + y^4 + \cos(y) = x^{17} - \sin x + y^2$

$$\frac{d}{dx}(3x^2 + y^4 + \cos(y)) = \frac{d}{dx}(x^{17} - \sin(x) + y^2)$$

$$6x + 4y^3\frac{dy}{dx} - (\sin(y))\frac{dy}{dx} = 17x^{16} - \cos(x) + 2y\frac{dy}{dx}$$

Get all terms with $\frac{dy}{dx}$ on 1 side; terms without $\frac{dy}{dx}$ on other side.

$$4y^3\frac{dy}{dx} - \frac{dy}{dx}\sin(y) - 2y\frac{dy}{dx} = 17x^{16} - \cos(x) - 6x$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(4y^3 - \sin(y) - 2y) = 17x^{16} - \cos(x) - 6x$$

$$\frac{dy}{dx} = \frac{17x^{16} - \cos(x) - 6x}{4y^3 - \sin(y) - 2y}$$

Example 4: Find $\frac{dy}{dx}$ for the equation $\frac{1}{y^4} - \frac{5}{x^4} = 1$.

$$y^{-4} - 5x^{-4} = 1$$

$$\frac{d}{dx}\left(y^{-4} - 5x^{-4}\right) = \frac{d}{dx}(1)$$

$$-4y^{-5}\frac{d}{dx}(y) + 20x^{-5} = 0$$

$$-\frac{4}{y^5}\frac{dy}{dx} + \frac{20}{x^5} = 0$$

$$-\frac{4}{y^5}\cdot\frac{dy}{dx} = -\frac{20}{x^5}$$

$$\frac{dy}{dx} = -\frac{20}{x^5}\left(-\frac{y^5}{4}\right)$$

$$= \frac{20y^5}{4x^5}$$

$$= \boxed{\frac{5y^5}{x^5}}$$

Example 5: Find $\frac{dy}{dx}$ for the equation $(x-y)^4 = y^2$.

$$\frac{d}{dx}(x-y)^4 = \frac{d}{dx}(y^2)$$

$$4(x-y)^3\frac{d}{dx}(x-y) = 2y\frac{dy}{dx}$$

$$4(x-y)^3\left(1 - \frac{dy}{dx}\right) = 2y\frac{dy}{dx}$$

$$4(x-y)^3 - 4(x-y)^3\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Example 6: Find $\frac{dy}{dx}$ for the equation $x + \cos(x^2y) = y$

$$\frac{d}{dx}\left(x + \cos(x^2y)\right) = \frac{d}{dx}(y)$$

$$1 - \sin(x^2y)\frac{d}{dx}(x^2y) = \frac{dy}{dx}$$

$$1 - \sin(x^2y)\left(x^2\cdot\frac{d}{dx}(y) + y\frac{d}{dx}(x^2)\right) = \frac{dy}{dx}$$

$$1 - \sin(x^2y)\left(x^2\frac{dy}{dx} + y(2x)\right) = \frac{dy}{dx}$$

$$1 - x^2\sin(x^2y)\frac{dy}{dx} - 2xy\sin(x^2y) = \frac{dy}{dx}$$

$$1 - 2xy\sin(x^2y) = \frac{dy}{dx} + x^2\sin(x^2y)\frac{dy}{dx}$$

$$1 - 2xy\sin(x^2y) = \frac{dy}{dx}\left(1 + x^2\sin(x^2y)\right)$$

see next page

$\frac{dy}{dx} = \boxed{\frac{1-2xy\sin(x^2y)}{1+x^2\sin(x^2y)}}$ equivalent to $\boxed{\frac{dy}{dx} = \frac{-1+2xy\sin(x^2y)}{-1-x^2\sin(x^2y)}}$

multiply by $\frac{-1}{-1}$

Finding higher-order derivatives using implicit differentiation:

To find the second derivative, denoted $\frac{d^2y}{dx^2}$, differentiate the first derivative $\frac{dy}{dx}$ with respect to *x*.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

1st derivative is

$\frac{d}{dx}(y) = \frac{dy}{dx}$

Example 7: Find $\frac{d^2y}{dx^2}$ for the equation $x^3 - 2x^2 = y$.

1st find $\frac{dy}{dx}$:

$\frac{d}{dx}(x^3 - 2x^2) = \frac{d}{dx}(y)$

$3x^2 - 4x = \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x)$

$= \boxed{6x - 4}$

Example 8: Find $\frac{d^2y}{dx^2}$ for the equation $xy^2 - y = 3$.

1st find $\frac{dy}{dx}$: $\frac{d}{dx}(xy^2 - y) = \frac{d}{dx}(3)$

product rule

$x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$

$x \cdot 2y\frac{d}{dx}(y) + y^2(1) - \frac{dy}{dx} = 0$

$2xy\frac{dy}{dx} + y^2 - \frac{dy}{dx} = 0$

$\frac{dy}{dx}(2xy - 1) = -y^2$

$\frac{dy}{dx} = -\frac{y^2}{2xy-1}$

See next page

Ex 8 cont'd:

$$\frac{dy}{dx} = \frac{-y^2}{2xy-1}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(2xy-1)\frac{d}{dx}(-y^2) - (-y^2)\frac{d}{dx}(2xy-1)}{(2xy-1)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(2xy-1)\left(-2y\frac{dy}{dx}\right) + y^2\left(2x\frac{dy}{dx} + y\frac{d}{dx}(2x) - 0\right)}{(2xy-1)^2}$$

$$= \frac{(2xy-1)\left(-2y\frac{dy}{dx}\right) + y^2\left(2x\frac{dy}{dx} + y\cdot 2\right)}{(2xy-1)^2}$$

$$= \frac{-4xy^2\frac{dy}{dx} + 2y\frac{dy}{dx} + 2xy^2\frac{dy}{dx} + 2y^3}{(2xy-1)^2}$$

$$= \frac{-2xy^2\frac{dy}{dx} + 2y\frac{dy}{dx} + 2y^3}{(2xy-1)^2}$$

Substitute $\frac{dy}{dx} = \frac{-y^2}{2xy-1}$

$$= \frac{-2xy^2\left(\frac{-y^2}{2xy-1}\right) + 2y\left(\frac{-y^2}{2xy-1}\right) + 2y^3 \cdot \frac{2xy-1}{2xy-1}}{(2xy-1)^2}$$

$$= \frac{2xy^4 - 2y^3 + 2y^3(2xy-1)}{(2xy-1)^3}$$

$$= \frac{2xy^4 - 2y^3 + 4xy^4 - 2y^3}{(2xy-1)^3} = \boxed{\frac{6xy^4 - 4y^3}{(2xy-1)^3} = \frac{d^2y}{dx^2}}$$

Example 9: Find the equation of the tangent line to the ellipse $4x^2 + 16y^2 = 64$ at the point $(2, \sqrt{3})$.

Find $\frac{dy}{dx}$:

$$\frac{d}{dx}\left(4x^2 + 16y^2\right) = \frac{d}{dx}(64)$$

$$8x + 32y\frac{dy}{dx} = 0$$

$$32y\frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{32y} = -\frac{x}{4y}$$

Find slope: $m = \left.\frac{dy}{dx}\right|_{x=2,\, y=\sqrt{3}} = -\frac{2}{4\sqrt{3}} = -\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{2\cdot 3} = -\frac{\sqrt{3}}{6}$

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{3} = -\frac{\sqrt{3}}{6}(x-2)$$

$$y - \sqrt{3} = -\frac{\sqrt{3}}{6}x + \frac{2\sqrt{3}}{6}$$

$$y = -\frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{3} + \sqrt{3}$$

$$y = -\frac{\sqrt{3}}{6}x + \frac{\sqrt{3}}{3} + \frac{3\sqrt{3}}{3}$$

Definition: Two curves are said to be *orthogonal* if at each point of intersection, their tangent lines are perpendicular.

$$\boxed{y = -\frac{\sqrt{3}}{6}x + \frac{4\sqrt{3}}{3}}$$

Example 10: Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $4x^2 + 9y^2 = 72$ are orthogonal.

Homework Qs

2.4 #29

$$g(x) = \left(\frac{x+5}{x^2+2}\right)^2 = (x+5)^2(x^2+2)^{-2}$$ use product rule

or $g(x) = \dfrac{(x+5)^2}{(x^2+2)^2}$ and use quotient rule

or $g'(x) = \underbrace{2\left(\frac{x+5}{x^2+2}\right)\frac{d}{dx}\left(\frac{x+5}{x^2+2}\right)}_{\text{chain rule}}$ and use quotient rule

2.4 #83

Determine points in interval $(0, 2\pi)$ at which graph of $f(x) = 2\cos x + \sin x$ has a horizontal tangent.

Find $f'(x)$ and set it equal to 0.

Solve the equation (it's a trig eqn)