2.6: Related rates

General idea for solving rate problems:

1. Draw a sketch if applicable. The only dimensions you put on your sketch should be those that do not change.
2. Write down, in calculus notation, the rates you know and want.
3. Write an equation relating the quantities that are changing.
4. Differentiate it implicitly, with respect to time.
5. Substitute known quantities.
6. Solve for the required rate.

Example 1: The radius of a sphere is increasing at the rate of 2 inches per minute. Find the rate of change in volume when the radius is 6 inches.

r = radius
t = time
V = volume

Know: $\frac{dr}{dt} = +\frac{2\text{in}}{\text{min}}$

Want: $\frac{dV}{dt}$

When: $r = 6\text{in}$

Key variables from Know and Want are r and V, Write an equation describing the relationship between r and V: $V = \frac{4}{3}\pi r^3$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

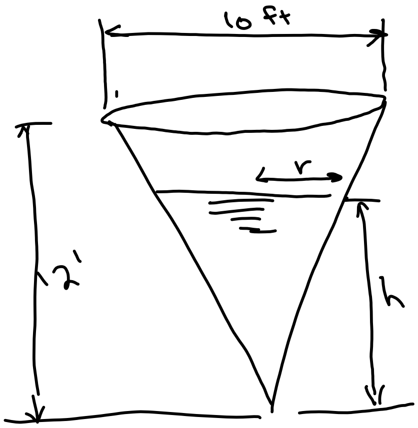
$$\left.\frac{dV}{dt}\right|_{r = 6\text{in},\ \frac{dr}{dt} = \frac{2\text{in}}{\text{min}}} = 4\pi (6\text{in})^2 \cdot \frac{2\text{in}}{\text{min}}$$

$$= \boxed{288\pi \ \text{in}^3/\text{min}}$$

$$\approx 904.8 \ \text{in}^3/\text{min}$$

(Example 1 cont'd)

Example 2: A conical tank (with vertex down) has a diameter of 10 feet at the top and is 12 feet deep. If water is draining out at 10 cubic feet per minute, what is the rate of change in depth when the water is 8 feet deep?



Know: $\frac{dV}{dt} = -10\frac{\text{ft}^3}{\text{min}}$

Want: $\frac{dh}{dt}$

When: $h = 8$ ft

Key Variables from Know & Want: h, V

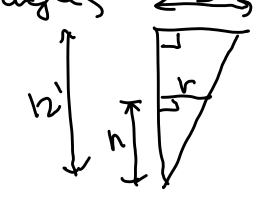
V = volume

Need eqn relating V and h:

Volume: $V = \frac{1}{3}\pi r^2 h$
of a cone

Want to get rid of r: need relationship between r and h

Similar triangles



$$\frac{5}{12} = \frac{r}{h} \quad \text{OR} \quad \frac{5}{r} = \frac{12}{h}$$

$$\frac{5h}{12} = r \qquad 5h = 12r$$

$$\frac{5h}{12} = r$$

$$V = \frac{1}{3}\pi r^2 h$$

Subst $r = \frac{5h}{12} \Rightarrow V = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h$

Quiz due Wed 10/12:

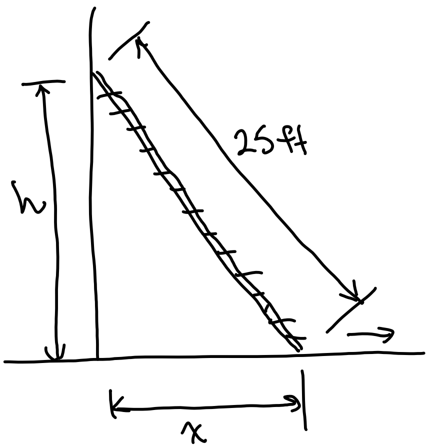
Finish this problem!

(look at Archive notes if you need help)

Warm-up problems for 2.5: 5, 11, 17, 21, 33, ~~45~~

(do these for Wed 10/12)

Example 3: A ladder 25 feet long is leaning against a wall. The base of the ladder is pulled away at 2 feet per second. How fast is the top of the ladder moving when the base is 9 feet away? What about when it is 24 feet away?



Know: $\frac{dx}{dt} = +2$ ft/sec

Want: $\frac{dh}{dt}$

When: $x = 9$ ft , $x = 24$ ft

Key variables from Know and Want: x, h

Write an eqn relating x and h to each other.

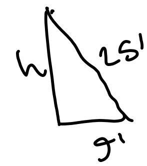
Pythagorean Theorem:

$x^2 + h^2 = (25\text{ft})^2$

$\frac{d}{dt}(x^2+h^2) = \frac{d}{dt}(625\text{ft}^2)$

$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

when $x = 9$ ft



$h^2 + 9^2 = 25^2$

$h^2 = 544$

$h = \sqrt{544}$ ft

$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

$2(9\text{ft})(2\text{ft/sec}) + 2(\sqrt{544}\text{ft})\frac{dh}{dt} = 0$

cont'd next page

Example 4: A particle is moving along the parabola $y^2 = 4x + 8$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. How fast is the x-coordinate changing at this instant?

Ex 3 cont'd

$$36\ \text{ft}^2/\text{sec} + 2\sqrt{544}\ \text{ft}\ \frac{dh}{dt} = 0$$

$$2\sqrt{544}\ \text{ft}\ \frac{dh}{dt} = -36\ \text{ft}^2/\text{sec}$$

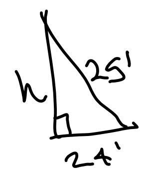
$$\frac{dh}{dt} = -\frac{\overset{18}{\cancel{36}}\ \text{ft}^{\cancel{2}}}{\text{sec}} \cdot \frac{1}{\underset{1}{\cancel{2}}\sqrt{544}\ \cancel{\text{ft}}}$$

$$= -\frac{18}{\sqrt{544}}\ \text{ft/sec}$$

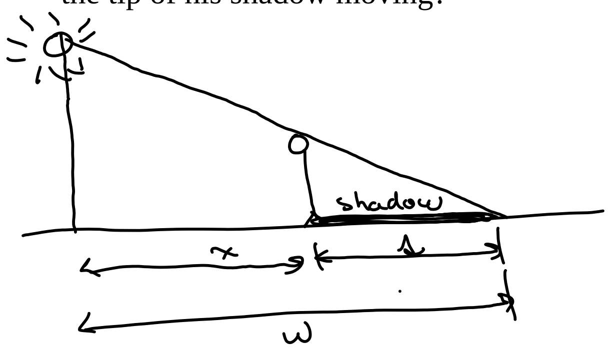
$$\approx \boxed{-0.7717\ \text{ft/sec}}$$

(when base is 9 ft from wall)

when base is 24 ft from wall:



Example 5: A 6-foot tall man walks away from a 30-foot tall lamppost at a speed of 400 feet per minute. When he is 50 feet away from the lamppost, at what rate is his shadow lengthening? How fast is the tip of his shadow moving?



Know: $\frac{dx}{dt} = \frac{400 \text{ ft}}{\text{min}}$

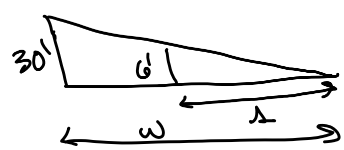
Want: $\frac{ds}{dt}$ and $\frac{dw}{dt}$

When: $x = 50$ ft

Write an eqn relating x, s, w. (Variables from the know rate and want rates)

$x + s = w$ Differentiating here would give us $\frac{dx}{dt}, \frac{ds}{dt}, \frac{dw}{dt}$, and we only know one ~~of~~ of these. Need to get rid of a variable.

Similar triangles: $\frac{s}{6'} = \frac{w}{30'} \Rightarrow s = \frac{6' w}{30'} = \frac{1}{5} w$



See next page

Example 6: At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport. The helicopter is rising vertically at a speed of 44 ft/second. How fast is the distance between the helicopter and the man changing when it is at an altitude of 120 ft?

Ex 5 cont'd

We have $x + s = w$.

We also have $s = \frac{1}{5} w$.

Substitute $s = \frac{1}{5} w$ into $x + s = w$.

$$x + \frac{1}{5} w = w$$

$$x = w - \frac{1}{5} w$$

$$x = \frac{4}{5} w$$

Differentiate: $\frac{d}{dt}(x) = \frac{d}{dt}\left(\frac{4}{5} w\right)$

$$\frac{dx}{dt} = \frac{4}{5} \frac{dw}{dt}$$

$$\frac{5}{4} \frac{dx}{dt} = \frac{dw}{dt}$$

$$\frac{dw}{dt} = \frac{5}{4} \cdot 400 \text{ ft/min} = 500 \text{ ft/min}$$

Go back to $s = \frac{1}{5} w$

Differentiate: $\frac{ds}{dt} = \frac{1}{5} \frac{dw}{dt}$

Substitute $\frac{dw}{dt} = 500$ ft/min: $\frac{ds}{dt} = \frac{1}{5} \cdot 500 \text{ ft/min} = 100 \text{ ft/min}$

The shadow is lengthening at 100 ft/min and the tip of the shadow is moving at 500 ft/min.