4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of *f* is a function whose derivative is *f*.

i.e. A function *F* is an antiderivative of *f* if $F'(x) = f(x)$.

Example 1: $F(x) = x^3 + 5x$ is an antiderivative of $f(x) = 3x^2 + 5$.

because $F'(x) = 3x^2 + 5$.

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$G(x) = x^3 + 5x + 7$

$H(x) = x^3 + 5x - 12$

or any constant C: $H_c(x) = x^3 + 5x + C$

So we have a whole "family" of antiderivatives of *f*.

> Definition: A function *F* is called an antiderivative of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

> Theorem: If *F* is an antiderivative of *f* on an interval *I*, then all antiderivatives of *f* on *I* will be of the form
>
> $$F(x) + C$$
>
> where *C* is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$H(x) = x^3 + 5x + C$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$\boxed{F(x) = x^6 + \sin x + C}$

Check: $F'(x) = 6x^5 + \cos x + 0$

$= 6x^5 + \cos x$ ✓

Integration:

Integration is the process of finding antiderivatives.

$\int f(x)\,dx$

$\int f(x)dx$ is called the *indefinite integral* of *f*.

$\int f(x)dx$ is the family of antiderivatives, or the most general antiderivative of *f*.

This means: $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$.

The *c* is called the *constant of integration*.
(or arbitrary constant)

Integral sign

"with respect to *x*"

$$\int f(x)dx$$

Integrand

Example 4: Find $\int 3x^2 + 5\,dx$.

$$\int (3x^2+5)\,dx = \boxed{x^3 + 5x + c}$$

Example 5: Find $\int 6x^5 + \cos x\,dx$.

$$\int (6x^5 + \cos x)\,dx = \boxed{x^6 + \sin x + C}$$

Example 6: Find $\int \sec^2 x\,dx$.

$$\int \sec^2 x\,dx = \boxed{\tan x + c}$$

Check: $\frac{d}{dx}(\tan x + c) = \sec^2 x$ ✓ ok

Rules for Finding Antiderivatives:

Notation in this table: *F* is an antiderivative of *f*, *G* is an antiderivative of *g*,

Function	Antiderivative
k	$kx+c$
$kf(x)$	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n for $n \neq -1$ (Power rule *)	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

* memorize

1. $\int k \; dx = kx + c$ (*k* a constant)

2. $\int x^n dx = \frac{x^{\cancel{n}\, n+1}}{n+1} + c \quad (n \neq -1)$ $\qquad \int x^n dx = \frac{x^{n+1}}{n+1} + c$

3. $\int k \; f(x) \; dx = k \int f(x) \; dx$ (*k* a constant) $\qquad$ Note:

4. $\int [f(x)+g(x)] dx = \int f(x) dx + \int g(x) dx$ $\qquad \frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + C$

5. $\int \cos x \, dx = \sin x + c$ $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$ $\Rightarrow \int \sec x \tan x \, dx = \sec x + C$

6. $\int \sin x \, dx = -\cos x + c$

$\frac{d}{dx}(\cot x) = -\csc^2 x \Rightarrow \frac{d}{dx}(-\cot x) = \csc^2 x$

be able to figure out

7. $\int \sec^2 x \, dx = \tan x + c$ $\qquad \Rightarrow \int \csc^2 x = -\cot x + c$

8. $\int \sec x \tan x \, dx = \sec x + c$ $\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

$\Downarrow$

$\frac{d}{dx}(-\csc x) = \csc x \cot x \Rightarrow \int \csc x \cot x \, dx = -\csc x + c$

$\int \csc x \cot x \, dx = -\csc x + c$

$\int \csc^2 x \, dx = -\cot x + c$

Example 7: Find the general antiderivative of $f(x) = \frac{1}{2}$.

$F(x) = \int \frac{1}{2} dx = \boxed{\frac{1}{2}x + C}$

Example 8: Find $\int x^3\,dx$.

$$\int x^3 dx = \frac{x^{3+1}}{3+1} + c = \boxed{\frac{x^4}{4} + c}$$

Check: $\frac{d}{dx}\left(\frac{x^4}{4}+c\right) = \frac{d}{dx}\left(\frac{1}{4}x^4 + c\right) = \frac{1}{4}(4x^3) + 0 = x^3$ ✓

Example 9: Find $\int 7x^2\,dx$.

$$\int 7x^2 dx = 7\int x^2 dx = 7\cdot\frac{x^3}{3} + c = \boxed{\frac{7x^3}{3} + c}$$

Check: $\frac{d}{dx}\left(\frac{7}{3}x^3\right) = \frac{7}{3}\cdot 3x^2 = 7x^2$ ✓

Example 10: Find $\int \frac{1}{x^5}\,dx$.

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + c = \frac{x^{-4}}{-4} + c = \boxed{-\frac{1}{4x^4} + c}$$

Check: $\frac{d}{dx}\left(-\frac{1}{4}x^{-4}\right) = -\frac{1}{4}(-4x^{-5}) = \frac{4}{4}x^{-5} = \frac{1}{x^5}$ ✓

Example 11: Find the general antiderivative of $f(x) = \frac{5}{x^2}$.

$$F(x) = \int \frac{5}{x^2} dx = \int 5x^{-2} dx = 5\int x^{-2} dx = 5\cdot\frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{5x^{-1}}{-1} + c = -\frac{5}{x} + c$$

$\boxed{F(x) = -\frac{5}{x} + c}$ is the general antiderivative of f.

Example 12: $\int (6x^2 - 3x + 9)\,dx$

$$\int (6x^2 - 3x + 9)\,dx = \frac{6x^3}{3} - 3\frac{x^2}{2} + 9x + c$$

$$= \boxed{2x^3 - \frac{3}{2}x^2 + 9x + c}$$

Check it!

Example 13: Find $\int 3\sqrt{x}\,dx$.

$$\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}} dx = 3\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = 3\cdot\frac{x^{3/2}}{3/2} + c$$

$$= \cancel{3}\cdot\frac{2}{\cancel{3}}x^{3/2} + c = 2x^{3/2} + c = \boxed{2\sqrt{x^3} + c}$$

Check: $\frac{d}{dx}\left(2x^{3/2} + c\right)$

$2\cdot\frac{3}{2}x^{1/2} = 3x^{1/2}$ ✓

Recall: $\int \sin x\,dx = -\cos x + c$

$\int \cos\,dx = \sin x + c$

Example 14: Find $\int (3\cos x + 5\sin x)\,dx$.

$$\int (3\cos x + 5\sin x)\,dx = 3\int \cos x\,dx + 5\int \sin x\,dx$$

$$= 3(\sin x + c_1) + 5(-\cos x + c_2)$$

$$= 3\sin x + 3c_1 - 5\cos x + 5c_2 = 3\sin x - 5\cos x + \underbrace{3c_1 + 5c_2}_{c}$$

$$= \boxed{3\sin x - 5\cos x + c}$$

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\,dx$.

$$\int \left(\frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\right) dx = \int \left(\frac{x^7}{x^5} - \frac{x^{1/3}}{x^5} + \frac{3x^2}{x^5}\right) dx$$

$$= \int (x^2 - x^{-14/3} + 3x^{-3})\,dx = \frac{x^3}{3} - \frac{x^{-11/3}}{-11/3} + 3\frac{x^{-2}}{-2} + c$$

$$= \boxed{\frac{x^3}{3} + \frac{3}{11}x^{-11/3} - \frac{3}{2x^2} + c}$$

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin\theta}{3}$.

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}}\right) dx$

Example 18: $\int (6y^2 - 2)(8y + 5)\,dy$

$$\int (48y^3 + 30y^2 - 16y - 10)\,dy$$

Differential equations:

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find *f*. This is an example of a differential equation.

(or, given $y' = x^2 - 7$, find y)

$$f(x) = \int f'(x)\,dx = \int (x^2 - 7)\,dx = \frac{x^3}{3} - 7x + C$$

$$\boxed{f(x) = \frac{x^3}{3} - 7x + C}$$

Check it by differentiating

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

$f(0)=3$ is called an initial condition. We use this to find the c.

$$f(x) = \int f'(x)\,dx = \int (3x^2 + 2\cos x)\,dx = 3\cdot\frac{x^3}{3} + 2\sin x + c$$
$$= x^3 + 2\sin x + c$$

Use the initial condition to find c:

We have $f(x) = x^3 + 2\sin x + c$

$f(0)=3 \Rightarrow f(0) = 0^3 + 2\sin(0) + c = 3$. Solve for c:

$$0 + c = 3$$
$$c = 3$$

Solution is: $\boxed{f(x) = x^3 + 2\sin x + 3}$

Check it: $f'(x) = 3x^2 + 2\cos x + 0$ ✓ok.

$f(0) = 0^3 + 2\sin(0) + 3 = 0 + 3 = 3$ ✓ok

Example 21: Suppose that $f''(x) = 2x^3 - 6x^2 + 6x$, $\underbrace{f'(2) = -1, \text{ and } f(-1) = 4}_{\text{Two initial conditions.}}$. Find $f(x)$.

$f''(x) = 2x^3 - 6x^2 + 6x$

$f'(x) = \int f''(x)\,dx = \int (2x^3 - 6x^2 + 6x)\,dx$

$= \frac{2x^4}{4} - \frac{6x^3}{3} + \frac{6x^2}{2} + c_1 = \frac{1}{2}x^4 - 2x^3 + 3x^2 + c_1$

$f'(2) = -1 \Rightarrow f'(2) = \frac{1}{2}(2)^4 - 2(2)^3 + 3(2)^2 + c_1 = -1$

$8 - 16 + 12 + c_1 = -1$

$4 + c_1 = -1$

$c_1 = -5$

$f'(x) = \frac{1}{2}x^4 - 2x^3 + 3x^2 - 5$

$f(x) = \int f'(x)\,dx = \int (\frac{1}{2}x^4 - 2x^3 + 3x^2 - 5)\,dx = \frac{1}{2}\cdot\frac{x^5}{5} - 2\cdot\frac{x^4}{4} + 3\frac{x^3}{3} - 5x + c_2$

$= \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + c_2$

$f(-1) = 4 \Rightarrow f(-1) = \frac{1}{10}(-1)^5 - \frac{1}{2}(-1)^4 + (-1)^3 - 5(-1) + c_2 = 4$

$-\frac{1}{10} - \frac{1}{2} - 1 + 5 + c_2 = 4$

$\Rightarrow -\frac{1}{10} - \frac{5}{10} + \frac{40}{10} + c_2 = 4$

$\frac{34}{10} + c_2 = 4$

$c_2 = 4 - \frac{34}{10}$

$= \frac{40}{10} - \frac{34}{10}$

$= \frac{6}{10} = \frac{3}{5}$

$$\boxed{f(x) = \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + \frac{3}{5}}$$

Example 22: Suppose that $f''(x) = 12x^2 - 18x$, $f(1) = 2$, and $f(-3) = 1$. Find $f(x)$.

$f'(x) = \int f''(x)\,dx = \int (12x^2 - 18x)\,dx = 12\frac{x^3}{3} - \frac{18x^2}{2} + c_1$

$f'(x) = 4x^3 - 9x^2 + c_1$

$f(x) = \int f'(x)\,dx = \int (4x^3 - 9x^2 + c_1)\,dx$

$= 4\cdot\frac{x^4}{4} - 9\cdot\frac{x^3}{3} + c_1x + c_2$

$f(x) = x^4 - 3x^3 + c_1x + c_2$

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time *t*, then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

Note: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2.

Example 23: Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

$$s(t) = \int v(t)\,dt = \int (2\sin t + \cos t)\,dt$$

$$= 2(-\cos t) + \sin t + c$$

$$= -2\cos t + \sin t + c$$

$s(0) = 3 \Rightarrow$ $s(0) = -2\cos(0) + \sin 0 + c = 3$

$$-2(1) + 0 + c = 3$$

$$-2 + c = 3$$

$$c = 5$$

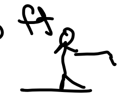
Position function is $\boxed{s(t) = -2\cos t + \sin t + 5}$

Example 24: Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?

$s(t)$ = position

Given: $s'(0) = +40$ ft/sec

$s(0) = 30$ ft

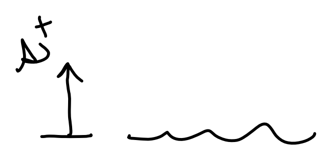


$$a(t) = s''(t) = -32 \text{ ft/s}^2$$

$$s'(t) = \int (-32)\,dt = -32t + c_1$$

Find c_1: $s'(0) = 40 \Rightarrow s'(0) = -32(0) + c_1 = 40$

$$c_1 = 40$$



$$s'(t) = -32t + 40$$

$$s(t) = \int s'(t)\,dt = \int (-32t + 40)\,dt = -32\frac{t^2}{2} + 40t + c_2$$

$$= -16t^2 + 40t + c_2$$

$s(0) = 30 \Rightarrow$ $s(0) = -16(0)^2 + 40(0) + c_2 = 30$

$$c_2 = 30$$

$$\boxed{s(t) = -16t^2 + 40t + 30}$$

Note: For acceleration due to gravity g and initial height h_0, and initial velocity v_0,

the position of an object in freefall (neglect air resistance) is $\boxed{s(t) = -\frac{1}{2} g t^2 + v_0 t + h_0}$

How high does it go?
when does it hit the water?

At max height: $s'(t) = 0$ (velocity = 0)

$$-32t + 40 = 0$$
$$40 = 32t$$
$$t = \frac{40}{32} = \frac{5}{4} = 1.25 \text{ sec}$$

max height is $s(1.25) = -16(1.25)^2 + 40(1.25) + 30 = \boxed{55 \text{ ft} \quad \text{max height}}$

when it hits the water, $s(t) = 0$:

$$-16t^2 + 40t + 30 = 0$$

Quadratic formula:
$$t = \frac{-40 \pm \sqrt{40^2 - 4(-16)(30}}{2(-16)}$$

$$\approx \frac{20 \pm \sqrt{880}}{16} \approx 3.104 \text{ sec}, -0.604 \text{ sec}$$

We want 3.104 seconds.

$\boxed{\text{It hits the water at } t = 3.104 \text{ seconds.}}$