3.1: Extrema on an Interval

Absolute maximum and minimum:

If $f(x) \le f(c)$ for every x in the domain of f, then $f(c)$ is the *maximum*, or *absolute maximum*, of f.

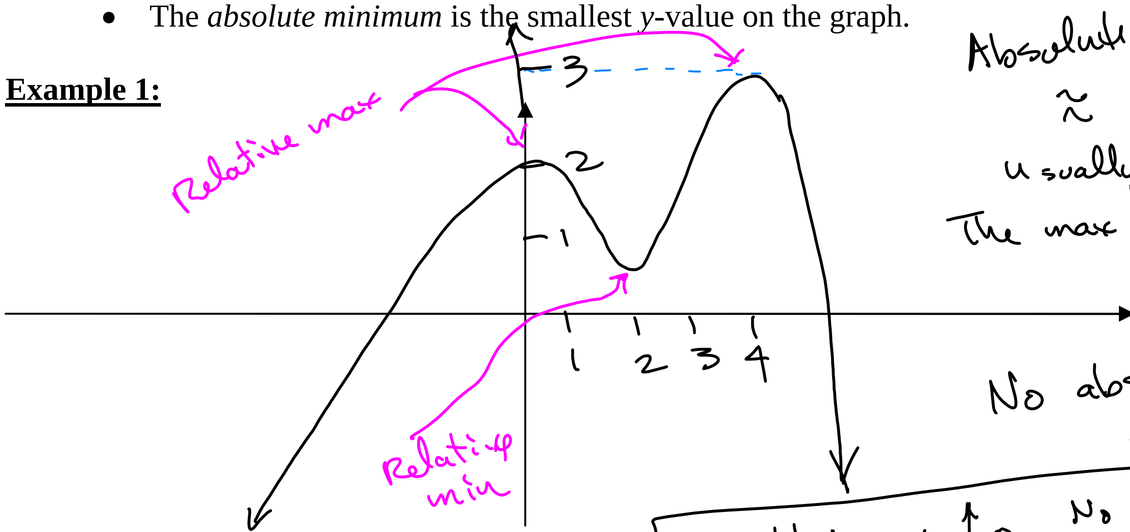
If $f(x) \ge f(c)$ for every x in the domain of f, then $f(c)$ is the *minimum*, or *absolute minimum* of f.

The maximum and minimum values of a function are called the *extreme values* of the function.

In other words,

- The *absolute maximum* is the largest y-value on the graph.
- The *absolute minimum* is the smallest y-value on the graph.

Example 1:



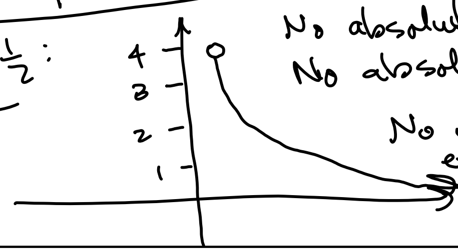
Absolute max: ≈ 3

usually write: The max is $f(4) = 3$

No absolute minimum.

Note: f is bounded above by 4 and below by 0.

Ex $1\frac{1}{2}$:



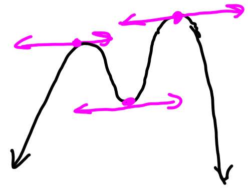
No absolute max,
No absolute min.
No relative extrema

Relative (Local) Maxima and Minima:

- A function f has a *relative maximum*, or *local maximum*, at $x = c$ if there is an interval (a,b) around c such that $f(x) \le f(c)$ for every x in (a,b). (These are the "hilltops").
- A function f has a *relative minimum*, or *local minimum*, at $x = c$ if there is an interval (a,b) around c such that $f(x) \ge f(c)$ for every x in (a,b). (These are the "bottoms of valleys").

In above example 1: Relative max: $f(0)=2$ and $f(3)=4$

Relative min: $f(2)=\frac{1}{2}$



Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

> Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. (tangent line is horizontal, so slope is 0)

This means that if f is differentiable at c and has a relative extreme at c, then the tangent line to f at c must be horizontal.

However, we must be careful. The fact that $f'(c) = 0$ (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at c.

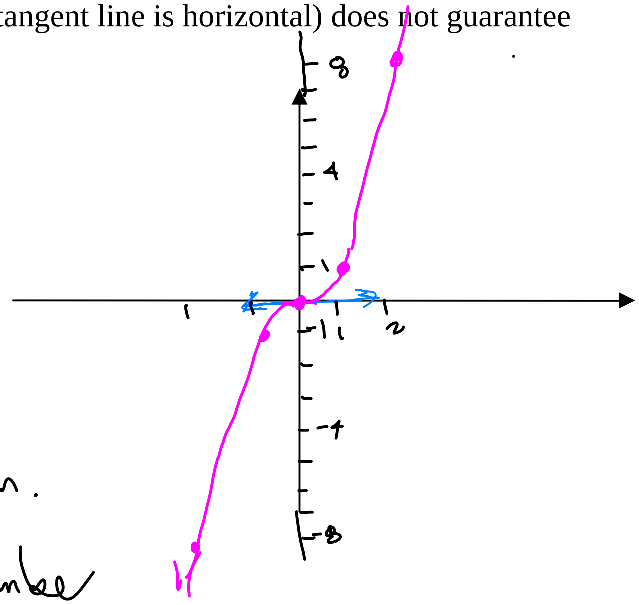
Example 2: $f(x) = x^3$

$f'(x) = 3x^2$

slope: $f'(0) = 3(0)^2 = 0$

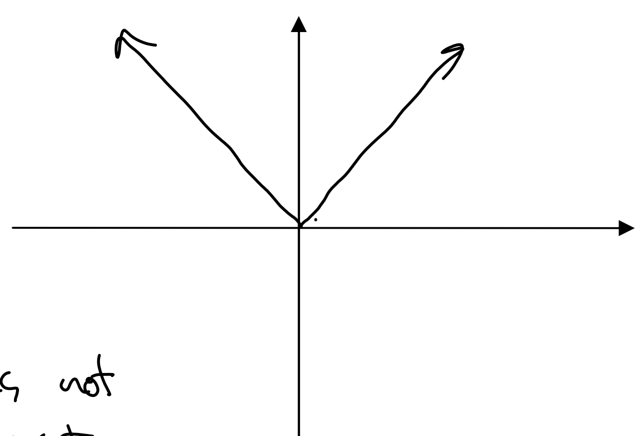
Horizontal tangent line, but no relative max nor min.

Note: 0 is a critical number



Example 3: There can be a local maximum or minimum at c even if $f'(c)$ does not exist.

$f(x) = |x|$



Absolute min $f(0) = 0$, (also a relative min), but $f'(0)$ does not exist

(sharp corner)

0 is a critical number

Critical numbers:

Critical Number: A *critical number* of a function *f* is a number *c* in the domain of *f* such that either $f'(c)=0$ or $f'(c)$ does not exist.

Theorem: If *f* has a local maximum or minimum at *c*, then *c* is a critical number of *f*.

Note: The converse of this theorem is not true. It is possible for *f* to have a critical number at *c*, but not to have a local maximum or minimum at *c*.

Example 4: Find the critical numbers of $f(x)=x^3+\frac{17}{2}x^2-6x+4$.

Find $f'(x)$ and set it equal to 0.

$f'(x) = 3x^2 + \frac{17}{2}(2x) - 6$

Set $f'(x)=0$: $0 = 3x^2+17x-6$

$0 = (3x-1)(x+6)$

$3x-1=0$ or $x+6=0$

$3x=1$ or $x=-6$

$x=\frac{1}{3}$ or $x=-6$

Critical Numbers: $\frac{1}{3}, -6$

Example 5: Find the critical numbers of $f(x)=x^{2/3}$

$f(x) = x^{2/3}$

Find $f'(x)$: $f'(x) = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3\sqrt[3]{x}}$

f' is never 0.

f' is undefined for $x=0$.

$f(x)=\sqrt[3]{x^2}$

Domain of f: $(-\infty,\infty)$

Critical number: 0

Note: To solve $f'(x)=0$:

$0=\frac{2}{3\sqrt[3]{x}}$

mult by $3\sqrt[3]{x}$, you get $0=2$

no sol'n

(The only way a fraction can be 0 is if its numerator is 0)

Example 6: Find the critical numbers of $f(x)=\frac{x^2}{x-3}$

Domain: $\{x \mid x \neq 3\}$

$(-\infty,3)\cup(3,\infty)$

$f'(x) = \frac{(x-3)(2x)-x^2(1)}{(x-3)^2} = \frac{2x^2-6x-x^2}{(x-3)^2}$

$= \frac{x^2-6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}$

Set $f'(x)=0$

$0=\frac{x(x-6)}{(x-3)^2}$

$0=x(x-6)$

$x=0, x=6$

$f'(x)=0$ for $x=0, x=6$

$f'(x)$ is undefined for $x=3$.

But 3 is not a critical number because it's not in the domain of f.

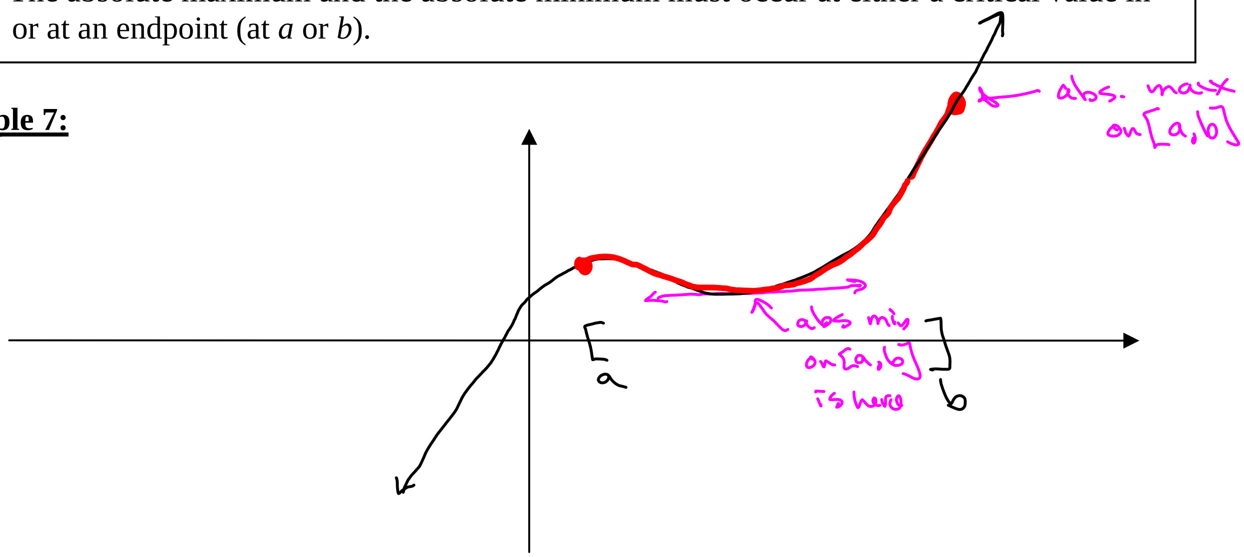
Critical Numbers: 0, 6

Absolute extrema on a closed interval:

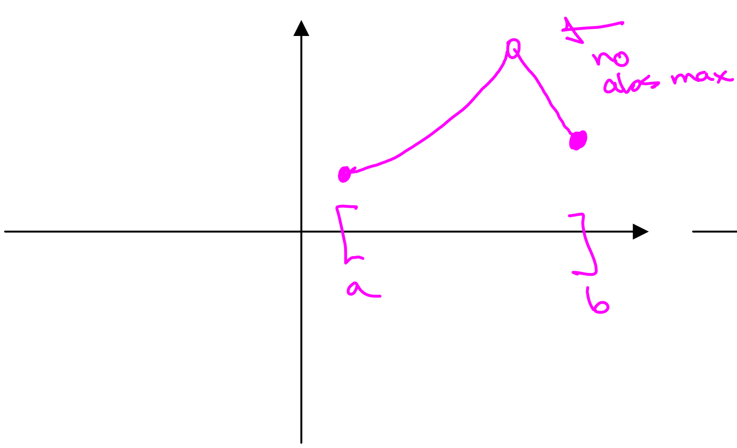
Extreme Value Theorem: If *f* is continuous on a closed interval $[a,b]$, then *f* has both an absolute maximum and an absolute minimum on $[a,b]$.

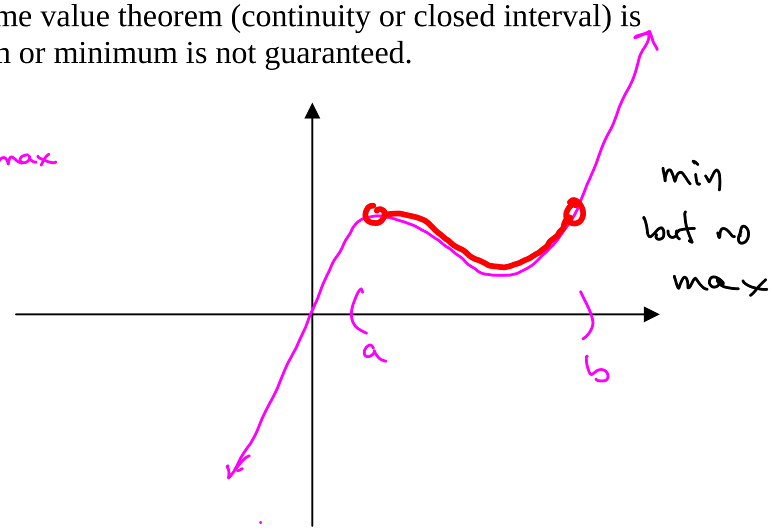
Note: The absolute maximum and the absolute minimum must occur at either a critical value in (a,b) or at an endpoint (at *a* or *b*).

Example 7:



Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.





<u>Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval</u>:

1. Find the critical values in (a,b).
2. Compute the value of f at each critical value in (a,b) and also compute $f(a)$ and $f(b)$.
3. The absolute maximum is the largest of these y-values and the absolute minimum is the smallest of these y-values.

<u>Example 9:</u> Find the absolute extrema for $f(x) = x^2 + 2$ on the interval $[-2,3]$.

f is continuous. Domain: $(-\infty,\infty)$

Find critical numbers: $f'(x) = 2x$
(critical values)

$0 = 2x$

$0 = x$

critical number: 0

$f(-2) = (-2)^2 + 2 = 6$

$f(3) = (3)^2 + 2 = 11 \leftarrow$ max

$f(0) = 0^2 + 2 = 2 \leftarrow$ min

$\boxed{\text{The absolute maximum is } f(3) = 11. \quad \text{The absolute minimum is } f(0) = 2}$

<u>Example 10:</u> Find the extreme values of $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ on the interval $[-2,1]$.

g is continuous, domain $(-\infty,\infty)$

$g'(x) = \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x$

$= 2x^3 - 2x^2 - 4x = 2x(x^2 - x - 2)$

$= 2x(x-2)(x+1)$

critical numbers: 0, 2, −1
(from setting $f'(x)=0$)

only 0 an −1 are in interval $[-2,1]$. So ignore 2 ($2 \notin [-2,1]$)

critical #s:

$g(0) = \frac{1}{2}(0)^4 - \frac{2}{3}(0)^3 - 2(0)^2 + 3 = 3$

$g(-1) = \frac{1}{2}(-1)^4 - \frac{2}{3}(-1)^3 - 2(-1)^2 + 3 = \frac{1}{2} + \frac{2}{3} - 2 + 3 = \frac{3}{6} + \frac{4}{6} + \frac{6}{6} = \frac{13}{6} = 2\frac{1}{6}$

endpts:

$g(-2) = \frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - 2(-2)^2 + 3 = 8 + \frac{16}{3} - 8 + 3 = \frac{25}{3} = 8\frac{1}{3}$

$g(1) = \frac{1}{2}(1)^4 - \frac{2}{3}(1)^3 - 2(1)^2 + 3 = \frac{1}{2} - \frac{2}{3} - 2 + 3 = \frac{3}{6} - \frac{4}{6} + \frac{6}{6} = \frac{5}{6}$

$\boxed{\text{Absolute max: } g(-2) = 8\tfrac{1}{3} \quad \text{Absolute min: } g(1) = \tfrac{5}{6}}$

Example 11: Find the absolute extrema of $h(x) = 6x^{2/3}$ on the intervals (a) $[-8,1]$, (b) $[-8,1)$, and. (c) $(-8,1)$.

Examples 11 and 12 are in archived notes from Spring 2015.

Example 12: Find the absolute maximum and absolute minimum of $f(x) = \sin 2x - x$ on the interval $[0, \pi]$.