3.2: Rolle's Theorem and the Mean Value Theorem

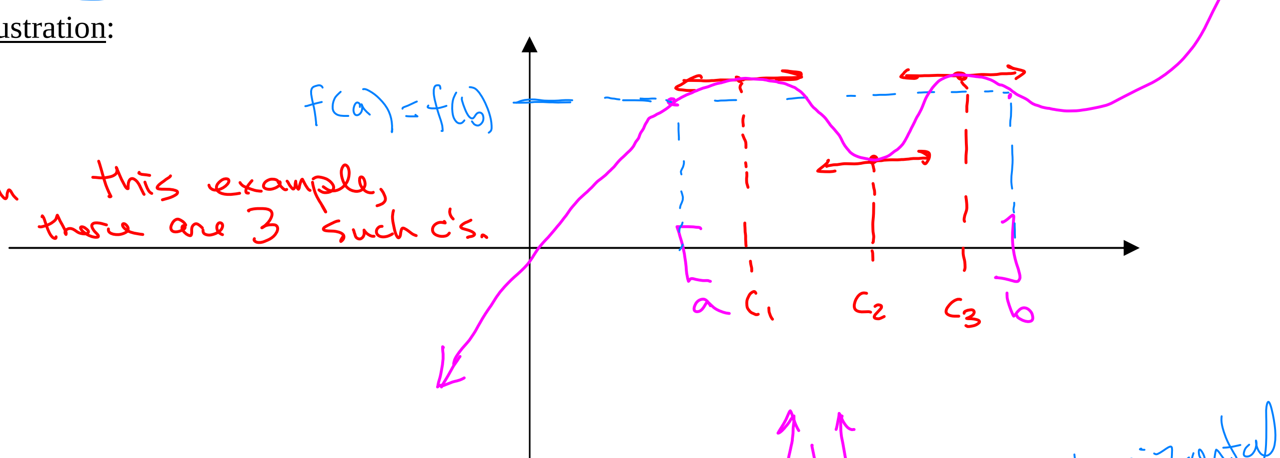
Rolle's Theorem: hypothesis of the theorem

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

If $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$.

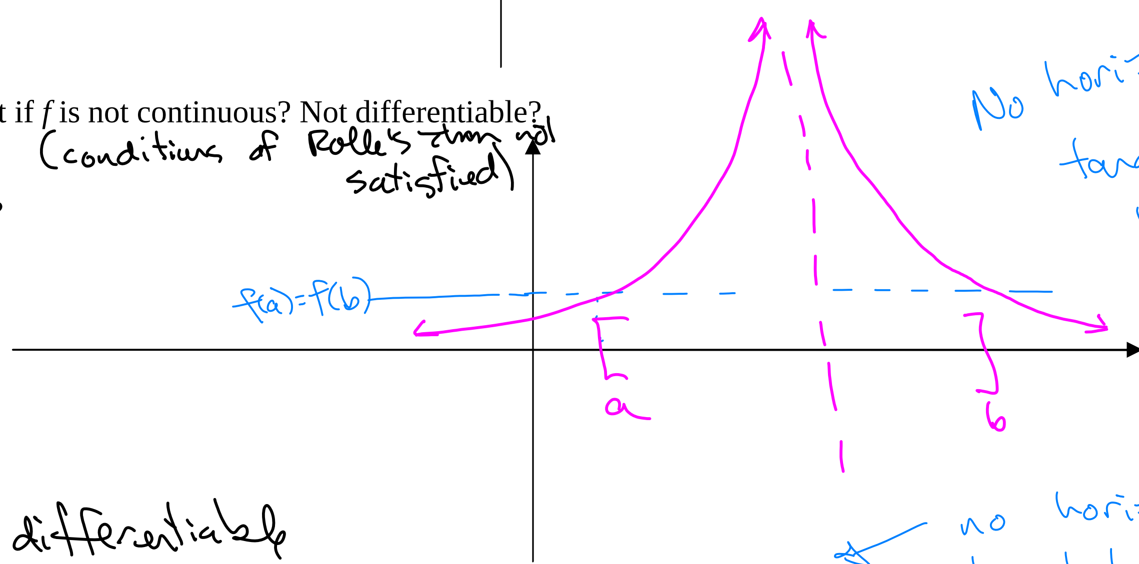
conclusion of the theorem

Illustration:



What if f is not continuous? Not differentiable? (conditions of Rolle's thm not satisfied)

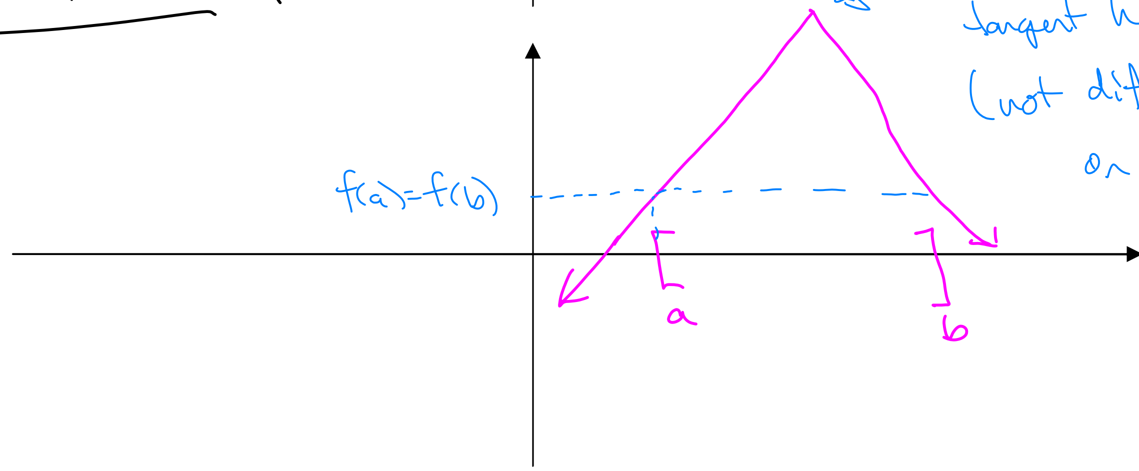
Not continuous



No horizontal tangents here.

Not differentiable

no horizontal tangent here (not differentiable on (a,b))



Example 1: Show that the function $f(x) = x^2 - 4x - 5$ satisfies the hypotheses of Rolle's Theorem on the interval $[-1, 5]$. Find all numbers c in $[-1, 5]$ that satisfy the conclusion of Rolle's Theorem.

hypotheses: f is continuous on $[-1,5]$
f is differentiable on $(-1,5)$
$f(-1) = f(5)$

f is a polynomial, so f is continuous and differentiable on $(-\infty, \infty)$.

$f(-1) = (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0$
$f(5) = 5^2 - 4(5) - 5 = 25 - 20 - 5 = 0$ } so $f(-1) = f(5)$.

therefore, Rolle's Thm guarantees there is a c in $(-1, 5)$ such that $f'(c) = 0$.

$f'(x) = 2x - 4$

set $f'(x) = 0$: $2x - 4 = 0$

$2x = 4$
$x = 2$

2 is the c that satisfies Rolle's.

Example 2: Show that the function $g(x) = -2x^4 + 16x^2$ satisfies the hypotheses of Rolle's Theorem on the interval $[-3, 3]$. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

g is continuous and differentiable on $(-\infty, \infty)$.

$g(-3) = -2(-3)^4 + 16(-3)^2 = -2(81) + 16(9) = -162 + 144 = -18$
$g(3) = -2(3)^4 + 16(3)^2 = -2(81) + 16(9) = -162 + 144 = -18$

So, Rolle's hypotheses are met.

$g'(x) = -8x^3 + 32x = 0$
$-8x(x^2 - 4) = 0$
$-8x(x+2)(x-2) = 0$

$x = 0, -2, 2$

All three are in our interval $[-3,3]$.

The c values are $0, -2, 2$.

<u>Mean Value Theorem</u>:

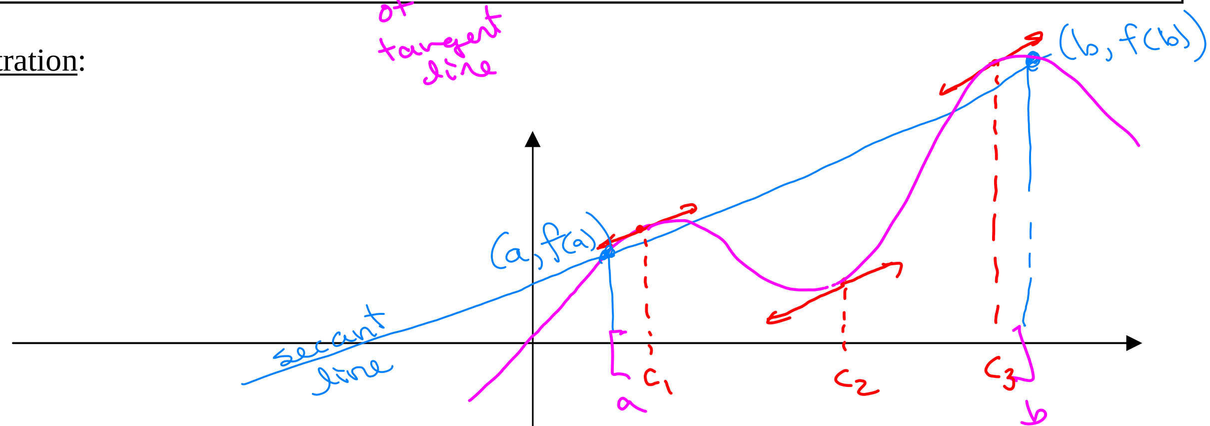
Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

slope of tangent line

slope of secant line

<u>Illustration</u>:



Slope of secant line: $\dfrac{f(b)-f(a)}{b-a}$

Slope of secant line: $f'(c)$

A few consequences of the Mean Value Theorem:

1) The Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line that contains the endpoints $(a, f(a))$ and $(b, f(b))$.

2) In terms of rates of change, the Mean Value Theorem guarantees that there is some point at which the <u>instantaneous</u> rate of change is equal to the <u>average</u> rate of change over $[a,b]$.

3) If the derivative of a function is 0 for every number in an interval, then the function is constant on that interval.

4) If two functions have the same derivative on an interval, they differ by a constant on that interval.

Example 3: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(x) = \sqrt{x} - 2x$ on the interval $[0, 4]$

Domain: $[0, \infty)$

f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$

Secant line goes through $(0, f(0))$ and $(4, f(4))$.

So $f(0) = \sqrt{0} - 2(0) = 0$

$f(4) = \sqrt{4} - 2(4) = 2 - 8 = -6$. So points are $(0,0)$ and $(4,-6)$.

Slope of secant line: $\dfrac{f(4) - f(0)}{4 - 0} = \dfrac{-6 - 0}{4 - 0} = -\dfrac{6}{4} = -\dfrac{3}{2}$

Find $f'(x)$: $f(x) = x^{1/2} - 2x$

$f'(x) = \frac{1}{2} x^{-1/2} - 2$

Set $f'(x)$ = slope of secant line: $\dfrac{1}{2\sqrt{x}} - 2 = -\dfrac{3}{2}$

$\dfrac{1}{2\sqrt{x}} = -\dfrac{3}{2} + 2$

$\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}$

$\dfrac{2\sqrt{x}}{1} = \dfrac{2}{1}$

$2\sqrt{x} = 2$

$\sqrt{x} = 1$

$x = 1^2 = 1$

1 is the number that satisfies the MVT

Example 4: As you drive by a Houston police car, your speed is clocked at 50 miles per hour. Five minutes and five miles further, you pass by another HPD officer, who clocks you at 55 miles per hour. If the speed limit on the entire stretch of highway is 55 mph, would the second officer be justified in writing you a speeding ticket?

Slope of secant line: $m = \dfrac{5 \text{ miles} - 0 \text{ miles}}{5 \text{ minutes} - 0 \text{ min}} = \dfrac{1 \text{ mile}}{\cancel{\text{min}}}\left(\dfrac{60 \cancel{\text{min}}}{1 \text{ hr}}\right)$

$= 60 \dfrac{\text{miles}}{\text{hr}}$ = average rate of change

From the mean value Thm, there must be an instant in this interval where the instantaneous velocity is equal to the avg velocity.

So yes, the driver deserves a ticket.