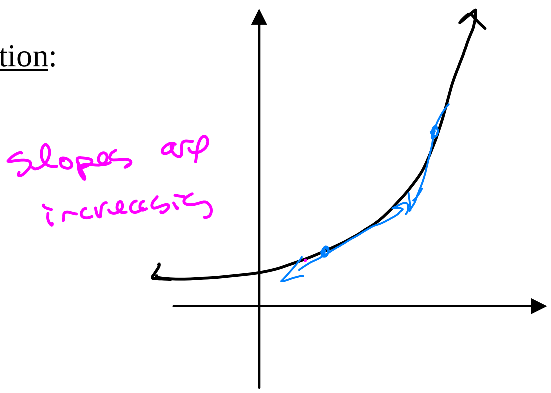
3.4: Concavity and the Second Derivative Test

Concavity:

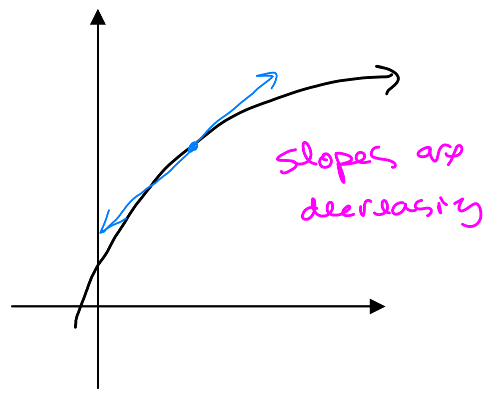
Definition:

- If the graph of f lies above all of its tangents on an interval, then it is called *concave upward* on that interval.
- If the graph of f lies below all its tangents on an interval, it is called *concave downward* on that interval.

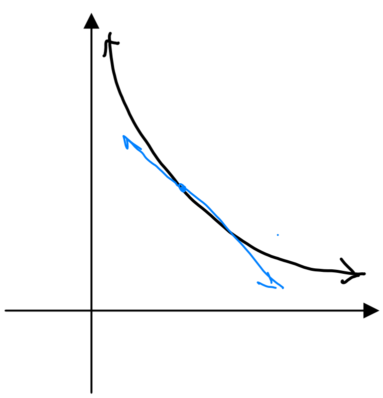
Illustration:



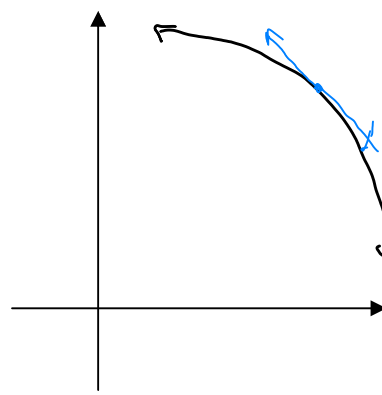
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



Decreasing and Concave Down

Notice the slopes of the tangent lines. When the curve is concave up, the slopes are increasing as you move from left to right.

When the curve is concave down, the slopes are decreasing as you move from left to right.

We find out whether f' is increasing or decreasing by looking at its derivative, which is f''.

Concavity Test:

$f''(x)>0$

- If $f''(x) > 0$ for all x in (a,b), then f is concave up on (a,b).
- If $f''(x) < 0$ for all x in (a,b), then f is concave down on (a,b).

$f''(x)<0$

Process for Determining Intervals of Concavity:

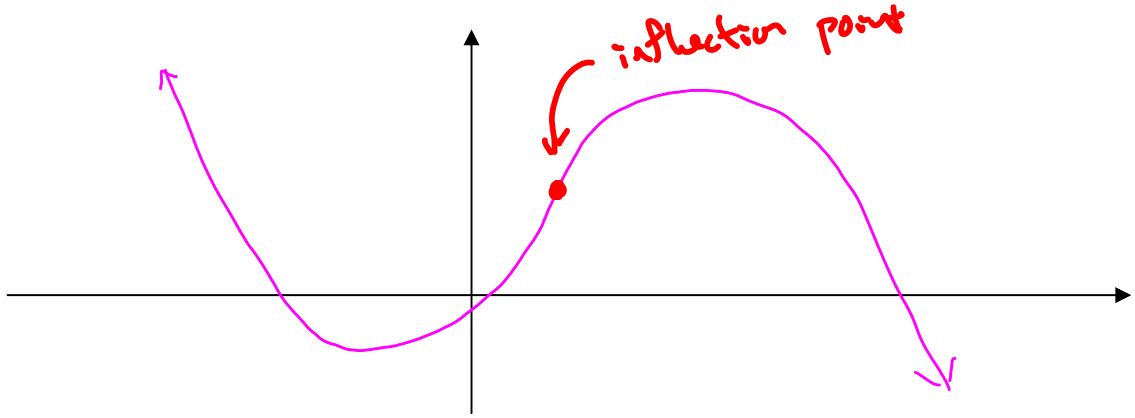
1. Find the values of x where $f''(x) = 0$ or where $f''(x)$ is not defined. Use these values of x to divide the number line into intervals.

2. Choose a test number c in each interval.
   - If $f''(c) > 0$, then f is concave up on that interval.
   - If $f''(c) < 0$, then f is concave down on that interval.

Inflection points:

An *inflection point* is a point on the graph of a function where the concavity changes.

Example 1:



Example 2: Find the intervals on which $f(x) = x^2$ is concave up and concave down.

$f'(x) = 2x$

$f''(x) = 2$ always positive

so f is concave up on $(-\infty, \infty)$

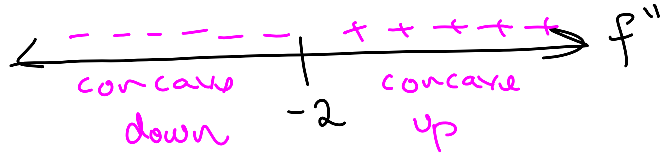
No inflection points.

Example 3: Determine the intervals of concavity and the inflection points of $f(x) = x^3 + 6x^2 - 36x + 18$. Domain: $(-\infty, \infty)$

$f'(x) = 3x^2 + 12x - 36$

$f''(x) = 6x + 12$
$= 6(x+2)$

set $f''(x) = 0$: $6x + 12 = 0$
$6x = -12$
$x = -2$



$(-\infty, -2)$: Test $x = -3$
$f''(-3) = 6(-3) + 12$
$= -18 + 12 = -6$
$(-)$

$(-2, \infty)$: Test $x = -1$
$f''(-1) = 6(-1) + 12 = 6$
$(+)$

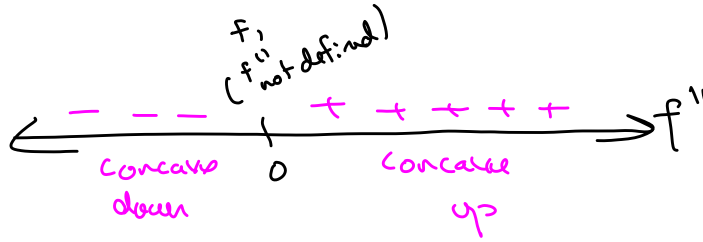
Inflection pt at $x = -2$.
Find y-value:

$f(-2) = (-2)^3 + 6(-2)^2 - 36(-2) + 18$
$= 106$

Inflection Pt: $(-2, 106)$
Concave up on $(-2, \infty)$,
Concave down on $(-\infty, -2)$.

Example 4: Determine the intervals of concavity and the inflection points of $f(x) = x + \frac{4}{x}$.

$f(x) = x + 4x^{-1}$
$f'(x) = 1 - 4x^{-2}$
$f''(x) = 8x^{-3} = \frac{8}{x^3}$



where is $f''(x) = 0$? never
(numerator never 0)

where is $f''(x)$ undefined? At 0
Is original f defined at 0? No

$(-\infty, 0)$: Test $x = -1$
$f''(-1) = \frac{8}{(-1)^3} = -8$ $(-)$

$(0, \infty)$: Test $x = 1$
$f''(1) = \frac{8}{1^3} = 8$ $(+)$

concave down on $(-\infty, 0)$
concave up on $(0, \infty)$

No inflection points

(because f is undefined at 0, you don't have an inflection pt at 0)

The second derivative test:

Notice: For a smooth (differentiable) function, the graph is concave upward at a relative minimum and concave downward at a relative maximum. local min → ← local max

Therefore, at a critical number, we can look at the sign of f'' to determine whether there is a relative minimum or relative maximum at that critical number.

> The Second Derivative Test (for Local Extremes):
>
> Suppose f'' is continuous near c.
> - If $f'(x)=0$ and $f''(c)<0$, then f has a relative maximum at c.
> - If $f'(x)=0$ and $f''(c)>0$, then f has a relative minimum at c.
> - If $f'(x)=0$ and $f''(c)=0$, then the test is inconclusive. Use the 1st derivative test instead.

Example 5: Use the second derivative test to find the local extremes of $f(x)=x^3+6x^2-36x+18$.

$f'(x)=3x^2+12x-36$

Find critical #s:

$0=3x^2+12x-36$

$0=3(x^2+4x-12)$

$0=3(x+6)(x-2)$

$x=-6,\ x=2$ critical #s: $-6, 2$

2nd derivative test:

$f''(x)=6x+12$

$f''(-6)=6(-6)+12=-24$ $(-)\Rightarrow$ local max at -6

$f''(2)=6(2)+12=24$ $(+)\Rightarrow$ local min at 2

(should find y-values too)

Example 6: Determine the local extremes of $f(x)=-2x^4+4x^3$

Find critical #s: $f'(x)=-8x^3+12x^2$

$=-4x^2(2x-3)$

critical values: $0, \frac{3}{2}$

see next page

2nd derivative test:

$f''(x)=-24x^2+24x$

$f''(0)=-24(0)^2+24(0)=0 \Rightarrow$ 2nd derivative is inconclusive test $\Rightarrow$ use 1st derivative test

Ex 6 cont'd:

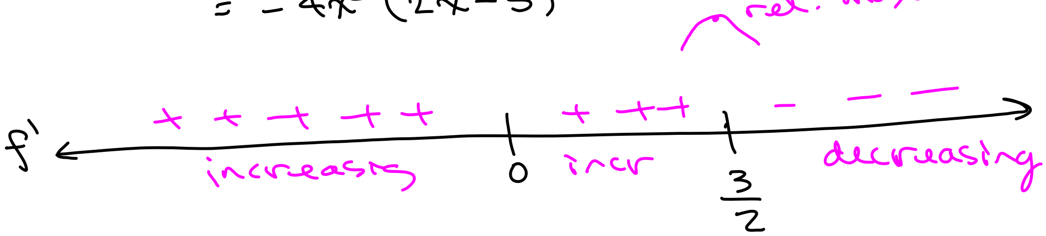
$$f'(x) = -8x^3 + 12x^2$$
$$= -4x^2(2x-3)$$

Critical numbers:

$x = 0$

Also $x = \frac{3}{2}$

$= 1\frac{1}{2}$



$(-\infty, 0)$: Test: $x = -1$

$$f'(-1) = -4(-1)^2(2(-1)-3)$$
$$= (-)(+)(-2-3)$$
$$= (-)(+)(-)$$
$$= (+)$$

$$f'(x) = -4x^2(2x-3)$$
$$\Rightarrow (-)(+)(2x-3)$$

$(0, \frac{3}{2})$: Test $x = 1$

$$f'(1) \Rightarrow (-)(+)(2(1)-3)$$
$$\Rightarrow (-)(+)(-) \Rightarrow (+)$$

We have a relative max at $\frac{3}{2}$

$(\frac{3}{2}, \infty)$: Test $x = 2$

$$f'(2) \Rightarrow (-)(+)(2(2)-3)$$
$$\Rightarrow (-)(+)(+) \Rightarrow (-)$$

Relative max is $f(\frac{3}{2}) =$