3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of *f*.
2. Find the *x*-intercepts and *y*-intercept, if any.
3. Determine the "end behavior" of *f*, that is, the behavior for large values of $|x|$ (limits at infinity). (this step can often be skipped)
4. Find the vertical, horizontal, and oblique asymptotes, if any.
5. Determine the intervals where *f* is increasing/decreasing.
6. Find the relative extremes of *f*, if any. (You should find both the *x*- and *y*-values.)
7. Determine the intervals where *f* is concave up/concave down.
8. Find the inflection points, if any. (You should find both the *x*- and *y*-values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.

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Find x-intercept:

Set $y=0$: $0 = x^3 - 6x^2 + 9x$

$= x(x^2 - 6x + 9)$

$= x(x-3)^2$

x-intercepts: 0, 3

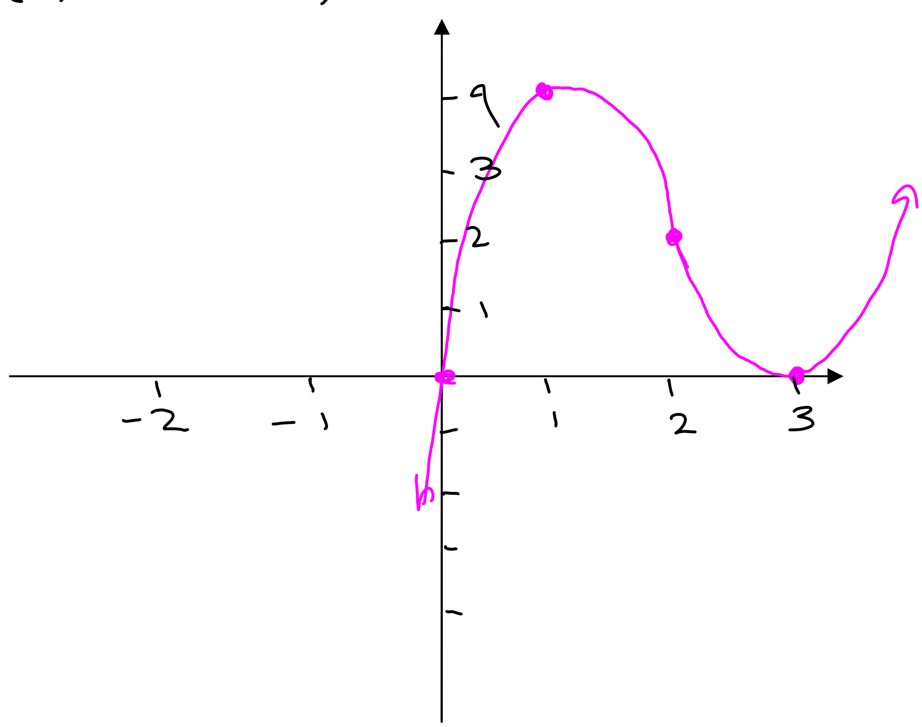
ordered pairs: (0,0) and (3,0)

Find y-intercept: set $x=0$.

$y = 0^3 - 6(0)^2 + 9(0)$

$y = 0$

ordered pair: (0,0)



Ex 1 cont'd

$$f(x) = x^3 - 6x^2 + 9x$$

$$f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x-1)(x-3)$$

critical #s: 1, 3

$(-\infty, 1)$: Test $x = 0$

$$f'(0) = 3(0-1)(0-3)$$
$$(+)(-)(-)$$
$$(+)$$

$(1, 3)$: Test $x = 2$

$$f'(2) = 3(2-1)(2-3)$$
$$(+)(+)(-)$$
$$(-)$$

$(3, \infty)$: Test $x = 4$

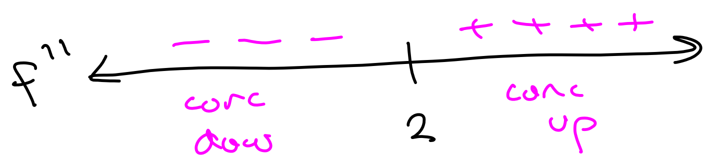
$$f'(4) = 3(4-1)(4-3)$$
$$(+)(+)(+)$$
$$(+)$$

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12 = 6(x-2)$$

or Set $f''(x) = 0$: $6x - 12 = 0$

$$6x = 12$$
$$x = 2$$



From f' sign chart, there's a relative min at 3 and a relative max at 1.

Find the y-values:

$$f(1) = 1^3 - 6(1)^2 + 9(1) = 4$$

$$f(3) = 3^3 - 6(3)^2 + 9(3) = 0$$

$\boxed{\text{Relative max at } (1,4) \quad \text{Relative min at } (3,0)}$

From f'' sign chart, there's an inflection pt at 2.

Find the y-value: $f(2) = 2^3 - 6(2)^2 + 9(2) = 2$

$\boxed{\text{Inflection pt: } (2,2)}$

Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

(Details in archive notes)

x-intercepts: $0, -4/3$

y-intercept: 0

f': $- - -$ | -1 | $+ + +$ | 1 | $+ + +$

decr, incr, incr

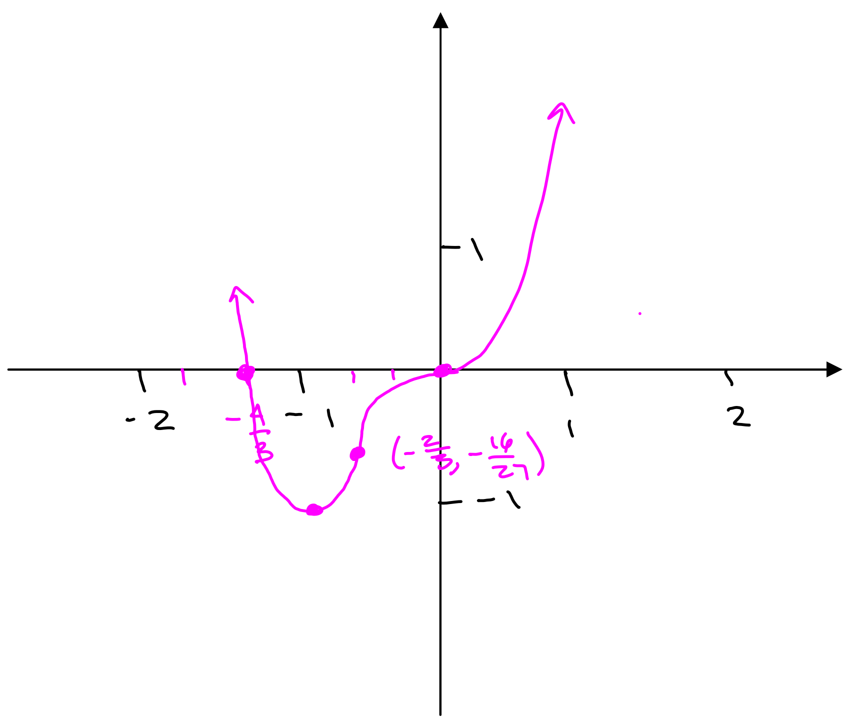
Relative min $(-1, -1)$.

f'': $+ + +$ | $-2/3$ | $- - -$ | 0 | $+ + +$

conc up, conc down, conc up

Inflection points: $(0, 0)$

$\left(-\frac{2}{3}, -\frac{16}{27}\right)$



Vertical asymptotes: $x=1$, $x=-1$

Horizontal asymptote: $y=0$

Domain f: all reals except 1, −1.

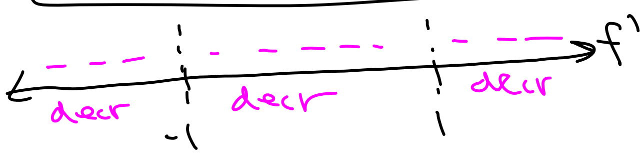
Example 3: Sketch the graph of $f(x)=\dfrac{2x}{x^2-1}$.

$$f'(x)=\frac{-2(x^2+1)}{(x^2-1)^2}$$

Where is $f'(x)$ zero? never

Where is $f'(x)$ undefined? ± 1

$\boxed{\text{No critical numbers}}$



For all x:

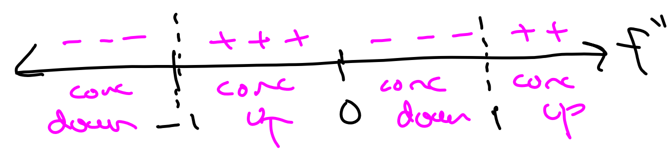
$$f'(x) \Rightarrow \frac{-2(+)}{(+)} \Rightarrow (-)$$

$\boxed{\text{Decreasing on } (-\infty,-1),\ (-1,1),\ \text{and } (1,\infty)}$

$$f''(x)=\frac{4x(x^2+3)}{(x^2-1)^3}$$

Where is f'' zero? At 0

Where is f'' undefined? ± 1 (same as for f)



$(-\infty,-1)$ Test $x=-2$

$$f''(-2)=\frac{4(-2)(+)}{((-2)^2-1)^3}=\frac{(-)}{(4-1)^3}$$

$$\Rightarrow \frac{(-)}{(+)^3}=\frac{(-)}{(+)}\Rightarrow(-)$$

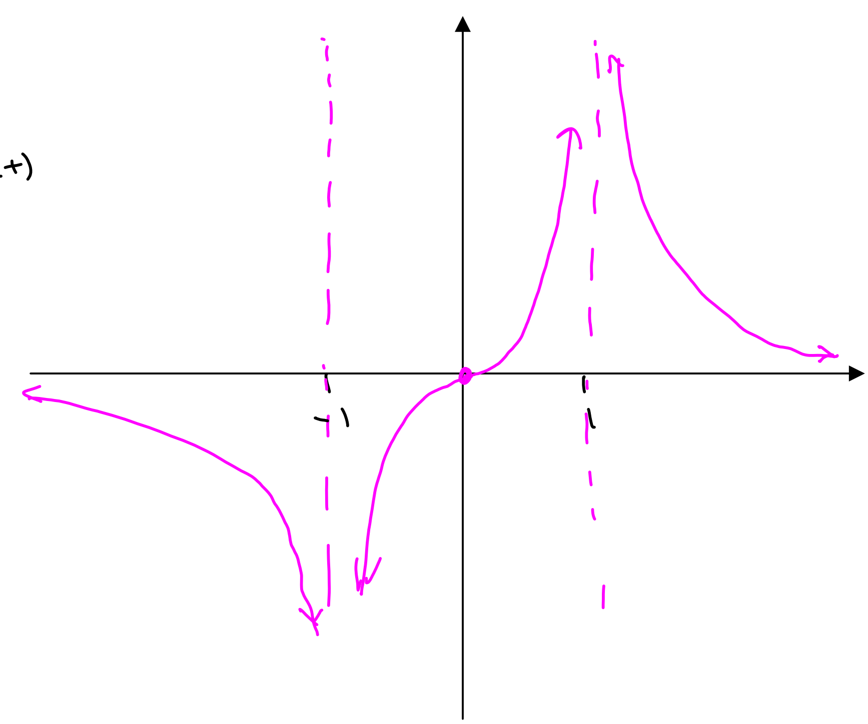
Testing f'' cont'd:

$(-1,0)$: Test $x=-0.5$

$$f''(-0.5)=\frac{4(-0.5)(+)}{((-.5)^2-1)^3}$$

$$\Rightarrow \frac{(-)}{(-)^3}\Rightarrow\frac{(-)}{(-)}\Rightarrow(+)$$

(Details in archive notes)



Check: Does it cross the horizontal asymptote?

set $y=1$: $\frac{x^2+1}{x^2-4}=1$

$x^2+1=x^2-4$

$1=-4$ no solution. No, doesn't cross the horizontal asymptote.

Example 4: Sketch the graph of $f(x)=\frac{x^2+1}{x^2-4}$.

$f(x)=\frac{x^2+1}{(x+2)(x-2)}$

$f'(x)=\frac{-10x}{(x^2-4)^2}$

$f''(x)=\frac{10(3x^2+4)}{(x^2-4)^3}$

Find x-intercepts: none (because numerator is never 0)

Vertical asymptote: $x=-2$, $x=2$

Find y-intercept: $f(0)=\frac{0^2+1}{0^2-4}=-\frac{1}{4}$

Find horizontal asymptote:

As $x\to\pm\infty$, $y\to\frac{1+\frac{1}{x^2}}{1-\frac{4}{x^2}}\to 1$

So horizontal asymptote is $y=1$.

1st derivative:

Where is $f'(x)=0$?

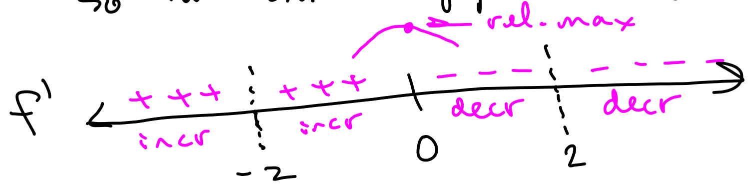
At $x=0$ (critical value)

Where is $f'(x)$ undefined?

At ± 2.

Neither of these are in domain of f.

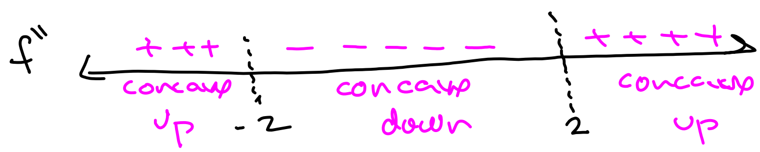
Note. $f'(x)\Rightarrow\frac{-10x}{(+)}$



2nd derivative $f''(x)=\frac{10(3x^2+4)}{(x^2-4)^3}$

Where is $f''(x)=0$? never

Where is $f''(x)$ undefined? ± 2 (same as f... these are vert. asymptotes)

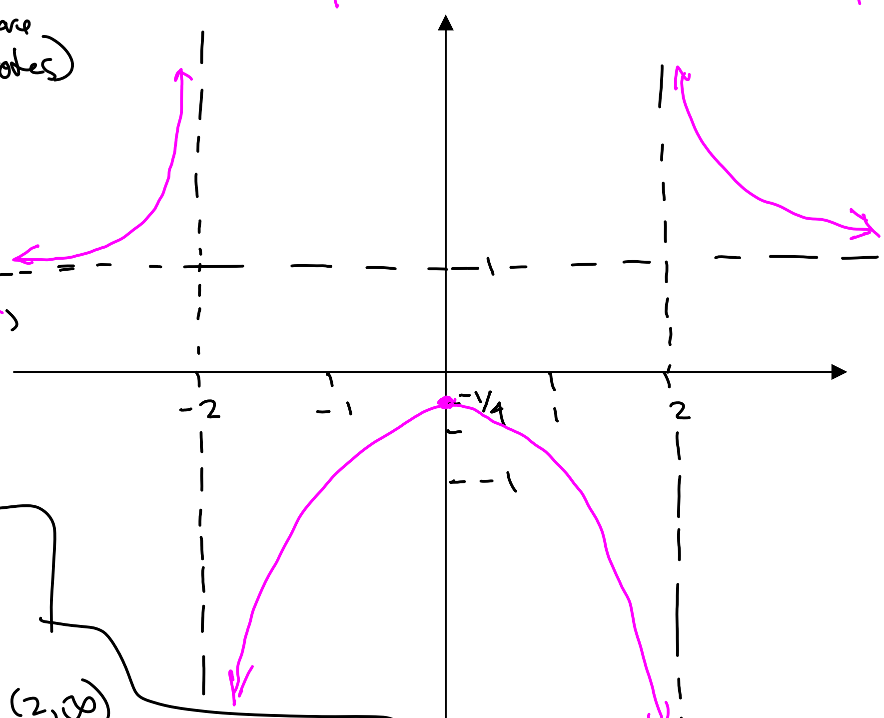


Note: $f''(x)\Rightarrow\frac{(+)}{(x^2-4)^3}$

$f''(-3)\Rightarrow\frac{(+)}{((-3)^2-4)^3}\Rightarrow\frac{(+)}{(9-4)^3}\Rightarrow\frac{(+)}{(+)}\Rightarrow(+)$

$f''(0)\Rightarrow\frac{(+)}{(0^2-4)^3}\Rightarrow\frac{(+)}{(-)^3}\Rightarrow\frac{(+)}{(-)}\Rightarrow(-)$

$f''(3)\Rightarrow\frac{(+)}{(3^2-4)^3}\Rightarrow\frac{(+)}{(+)^3}\Rightarrow(+)$



increasing on $(-\infty,-2)$ and $(-2,0)$
decreasing on $(0,2)$ and $(2,\infty)$
concave down on $(-2,2)$
concave up on $(-\infty,-2)$ and $(2,\infty)$
Relative max at $(0,-1/4)$. No inflection pts

Example 5: Sketch the graph of $f(x) = \dfrac{x^2-4}{x+3}$.

$$f(x) = \frac{(x+2)(x-2)}{x+3}$$

Vertical asymptote: $x = -3$

x-intercepts: $2, -2$

or $(2,0)$, $(-2,0)$

y-intercept: $-\frac{4}{3}$

or $(0, -\frac{4}{3})$

Find horizontal asymptote:

As $x \to \infty$, $y \to \dfrac{\frac{x^2}{x} - \frac{4}{x}}{\frac{x}{x} + \frac{3}{x}} \to \dfrac{x - \frac{4}{x}}{1 + \frac{3}{x}} \to \infty$

As $x \to -\infty$, $y \to -\infty$

no horizontal asymptote.

deg (num) = deg (denom) +1, so there is a slant asymptote. Find it:

$$\begin{array}{r} x - 3 \\ x+3 \overline{) x^2 + 0x - 4} \\ -(x^2 + 3x) \\ \hline -3x - 4 \\ -(-3x - 9) \\ \hline 5 \end{array}$$

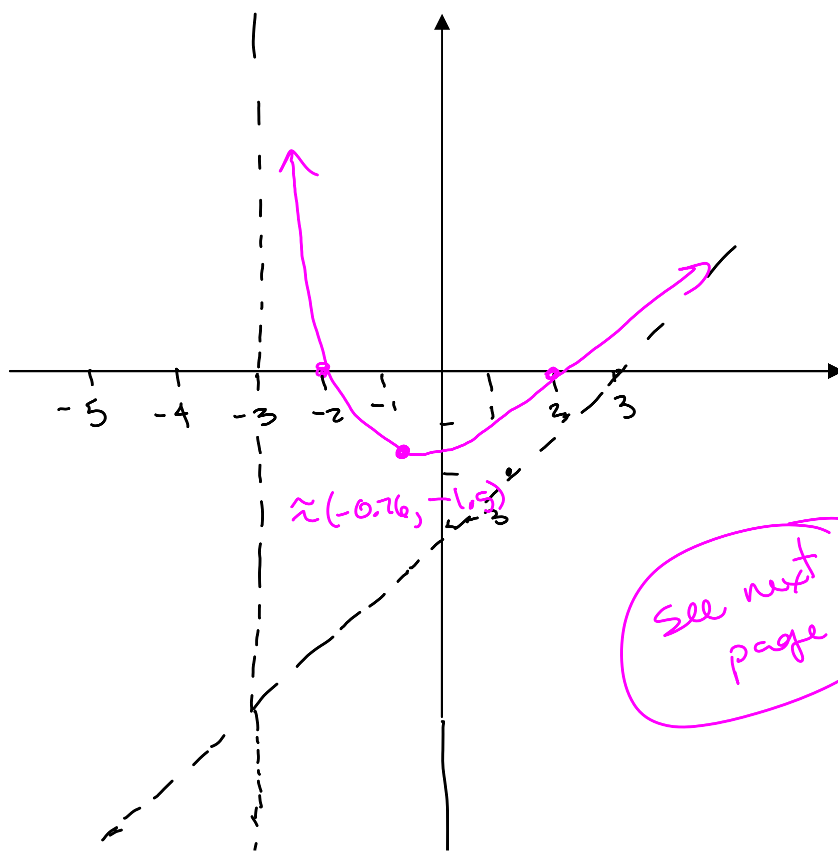
so $f(x) = \dfrac{x^2-4}{x+3} = x - 3 + \dfrac{5}{x+3}$

As $x \to \pm\infty$, $\dfrac{5}{x+3} \to 0$

so $f(x)$ approaches $y = x - 3$

$\boxed{\text{slant asymptote: } y = x - 3}$

$m = 1$
$b = -3$



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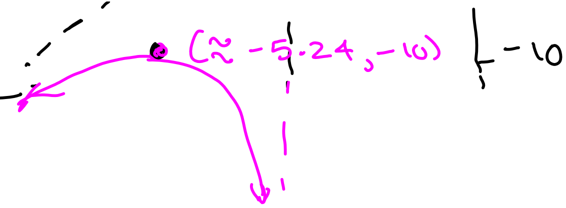
Example 5 cont'd



$$f(x) = \frac{x^2-4}{x+3}$$

$$f'(x) = \frac{(x+3)(2x)-(x^2-4)(1)}{(x+3)^2} = \frac{2x^2+6x-x^2+4}{(x+3)^2} = \frac{x^2+6x+4}{(x+3)^2}$$

Find critical numbers: Set numerator $=0$.

$$x^2+6x+4=0$$

Quadratic formula:
$$x = \frac{-6 \pm \sqrt{6^2-4(1)(4)}}{2(1)} = \frac{-6\pm\sqrt{20}}{2} = \frac{-6\pm 2\sqrt{5}}{2}$$
$$= \frac{2(-3\pm\sqrt{5})}{2} = -3\pm\sqrt{5}$$
$$\approx -0.763, \ -5.236$$

Find y-values:
$$f(-3+\sqrt{5}) \approx -1.527 \Rightarrow \text{Pairs } (-0.763, -1.527)$$
$$f(-3-\sqrt{5}) \approx -10.472 \qquad (-5.236, -10.472)$$

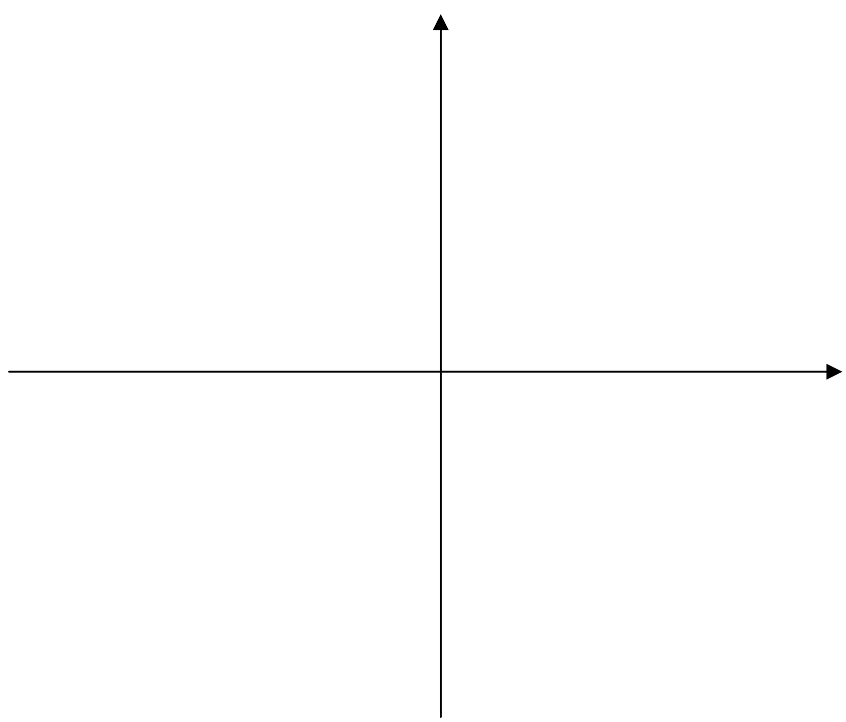
$$f'(x) = \frac{x^2+6x+4}{(x+3)^2}$$

$$f''(x) = \frac{(x+3)^2(2x+6)-(x^2+6x+4)(2)(x+3)(1)}{(x+3)^4}$$

$$= \frac{(x+3)^2(2x+6)-2(x^2+6x+4)(x+3)}{(x+3)^4} = \frac{(x+3)\left[(x+3)(2x+6)-2(x^2+6x+4)\right]}{(x+3)^4}$$

$$= \frac{2x^2+12x+18-2x^2-12x-8}{(x+3)^3} = \frac{10}{(x+3)^3}$$

Example 6: Sketch the graph of $f(x) = 5x^{2/3} - x^{5/3}$.



Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.