3.7: Optimization Problems

We often need to solve problems involving optimization: finding the maximum or minimum of some quantity.

Process for Solving Optimization Problems

1. Assign a variable to each quantity mentioned. If possible, draw and label a diagram.
2. Write an expression for the quantity to be optimized.
3. Write the quantity to be optimized as a function of one variable. Determine its domain.
4. Find the minimum or maximum by sketching the curve and finding the relative extrema, or by calculating the absolute maximum or minimum on a closed interval.

Example 1: Find two positive numbers such that the sum of the first and twice the second is 100, and their product is a maximum.

x = first number
y = 2nd number

Maximize P

Write an equation for the product P: $P = xy$

Need to get rid of either x or y:
Need an equation/relationship between x and y!

$$x + 2y = 100$$

solve for x: $x = 100 - 2y$

Substitute $x = 100 - 2y$ into $P = xy$

$P = (100-2y)(y)$
$P(y) = 100y - 2y^2$
$P'(y) = 100 - 4y$
Set $P'(y) = 0$: $100 - 4y = 0$
$100 = 4y$
$25 = y$

2nd deriv test: $P''(y) = -4$
conc down
so rel. max at 25

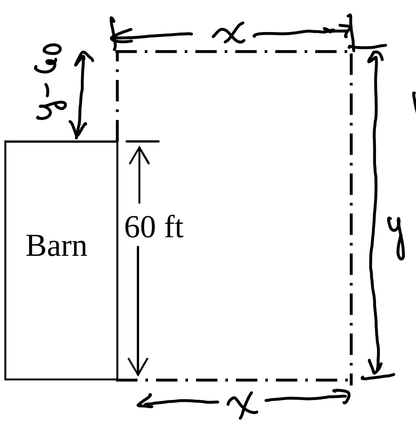
or, note that it's a quadratic opening down, so this must be the absolute max.

Find the 2 positive numbers: we have $y = 25$. Put this into
$x = 100 - 2y$
$x = 100 - 2(25) = 100 - 50$
$= 50$

The two numbers are 25 and 50

The maximum product is $25(50)$
$= 1250$.

Example 2: A farmer wants to construct a rectangular pen next to a barn 60 feet long, using the entirety of one side of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if 250 feet of fencing material is available.



Maximize: Area = A

$A = xy$

Need to get rid of a variable

$2x + y + y - 60 = 250$
$2x + 2y = 310$
Solve for x: $2x = 310 - 2y$
$x = \frac{310 - 2y}{2} = 155 - y$

See next page

Ex 2 cont'd:

$A = xy$

substitute $x = 155 - y$:

$A = (155 - y)y$

$A(y) = 155y - y^2$

$A'(y) = 155 - 2y = 0$

$155 = 2y$

$y = \frac{155}{2} = 77.5$

$x = 155 - 77.5 = 77.5$

The dimensions of the pen are
77.5 ft $\times$ 77.5 ft

Example 3: A rectangular piece of cardboard can be turned into an open box by cutting away squares from the corners and turning up the flaps. If a piece of cardboard is 6 inches wide and 16 inches long, find the dimensions of the box with maximum volume.

Maximize: Volume = V

Worked out during class - see Archive notes Fall 2014

Example 4: A dog food company decides to package its new dog treats, *Dusty's Yummy Doggy Kibbles*, in cylindrical cans. Each can will be filled to the top with 54 cubic inches of delicious dog treats. What height and radius should be used to minimize the amount of metal required?

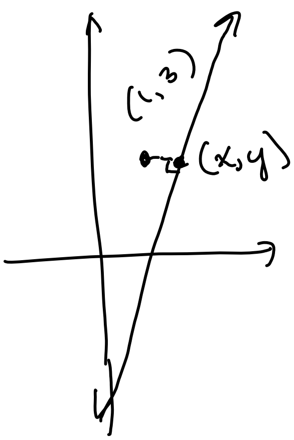
Minimize: Surface Area

Worked out during class - see Archive notes Fall 2014

Example 5: An isosceles right triangle has hypotenuse of length 2. What is the maximum area of a rectangle that can be inscribed in this triangle, if one side lies along the hypotenuse?

See Fall 2014 notes in archives

Example 6: Find the point on the line $y = 4x - 3$ that is closest to the point (1, 3).



minimize: Distance between (1,3) and (x,y)

$D = \text{Distance} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

when D is small, D^2 is small also.

It's easier to work with D^2 than D.

$D^2 = (x_2-x_1)^2 + (y_2-y_1)^2$

Using our points: $D^2 = (x-1)^2 + (y-3)^2$

want it in 1 variable only: $\Rightarrow D^2 = (x-1)^2 + (4x-3-3)^2$

substitute $y = 4x-3$ $\qquad = (x-1)^2 + (4x-6)^2$

Let $Z = D^2$: $Z = (x-1)^2 + (4x-6)^2$

$= x^2 - 2x + 1 + 16x^2 - 48x + 36$

$= 17x^2 - 50x + 37$

$Z' = 34x - 50$

next page

Closest pt is
$\left(\frac{25}{17}, \frac{49}{17}\right)$

$34x - 50 = 0$
$34x = 50$
$x = \frac{50}{34} = \frac{25}{17}$. $y = 4\left(\frac{25}{17}\right) - 3 = \frac{49}{17}$

Example 7: A closed rectangular box is to have a square base and a volume of 20 cubic feet. The material for the base costs 30 cents per square foot, and the material for the sides costs 10 cents per square foot, and the material for the top costs 20 cents per square foot. Determine the dimensions of the box which minimize cost. What is that minimum cost?

See Fall 2014 notes in Archives