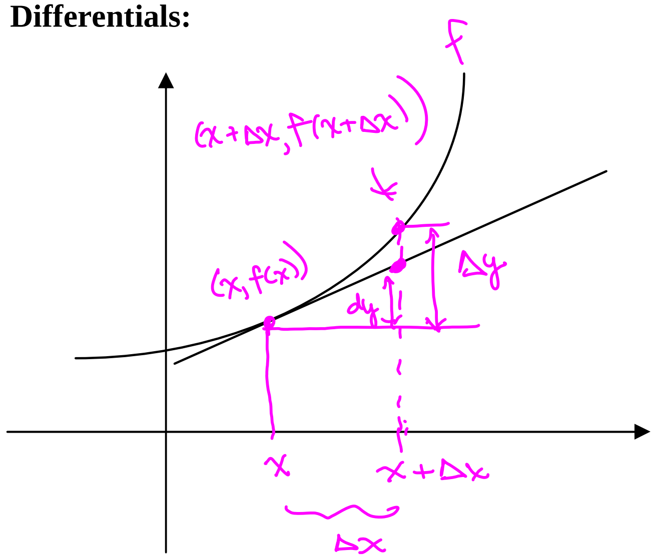
3.9: Differentials

$\Delta y = f(x+\Delta x) - f(x)$

Differentials:



Let $dx = \Delta x$

If $y = f(x)$ is a differentiable function, we can let dx represent an amount of change in x.

Then the *differential* dy is defined to be $dy = f'(x)dx$. $\Rightarrow$ $\frac{dy}{dx} = f'(x)$

dy is an approximation to $\Delta y = f(x+\Delta x) - f(x)$, which is the actual change in y.

Example 1: Find the differential dy for $y = \sqrt{6+x}$. Evaluate it when $x = 10$ and $dx = -0.3$. Compare it to the exact value of Δy.

$y = (6+x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(6+x)^{-1/2}(1) = \frac{1}{2\sqrt{6+x}}$

$\boxed{dy = \frac{1}{2\sqrt{6+x}}\,dx}$

when $x=10$ and $dx = -0.3$,

$dy = f'(x)\,dx$

$f'(10) = \frac{1}{2\sqrt{6+10}} = \frac{1}{2(4)} = \frac{1}{8}$

$dy = f'(x)\,dx = \frac{1}{8}(-0.3) = \boxed{-0.0375}$

If x changes by -0.3, y changes by approximately -0.0375.

Find the exact change in y:

$\Delta y = f(10-0.3) - f(10)$

$= f(9.7) - f(10)$

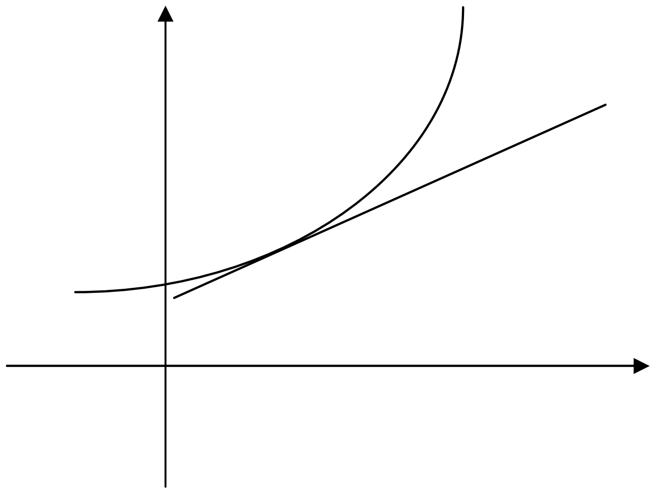
$= \sqrt{6+9.7} - \sqrt{6+10}$

$= \sqrt{15.7} - \sqrt{16}$

$\approx \boxed{-0.037677}$ actual change in y

Example 2: Compute dy and Δy if $y = 5x + x^3$ as x changes from 3 to 3.05.

The linearization of a function:



We use the tangent line to approximate the function

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

$$f(x) - f(a) \approx f'(a)(x - a)$$

$$f(x) \approx f(a) + f'(a)(x - a)$$

This tangent line gives us an approximation for the value of the function near *a*.

> The *linearization* of *f* at *a* is
>
> $$L(x) = f(a) + f'(a)(x - a)$$
>
> The approximation $f(x) \approx L(x)$ is the standard linear approximation of *f* at *a*.

Example 3: Find the linearization of $f(x) = x^4$ at 3. Use this to approximate $(3.013)^4$ and $(2.999)^4$.

→ equation of tangent line

$f'(x) = 4x^3$

Slope: $m = f'(3) = 4(3)^3 = 108$

Find the y-value: $f(3) = 3^4 = 81$

Point: $(3, 81)$, slope $= 108$

$y - y_1 = m(x - x_1)$

$y - 81 = 108(x - 3)$

$y = 108x - 324 + 81$

$y = 108x - 243$

Linearization:
$L(x) = 108x - 243$

$(3.013)^4 \approx L(3.013)$

$\approx 108(3.013) - 243$

$= \boxed{82.404}$

<u>Example 4:</u> Find the linearization of $f(x) = \sin x$ at $60°$. Use this to approximate $\sin 62°$ and $\sin 58°$.

Error propagation:

If x is the measured value of a variable and $x + \Delta x$ is the exact value of the variable, then Δx is the measurement error. If we use the measured value of x to calculate the value of a function f, then the <u>propagated error</u> is $\Delta y = f(x + \Delta x) - f(x)$. The propagated error can be estimated by calculating $dy = f'(x)dx \approx f'(x)\Delta x$.

<u>Estimating propagated error</u>:

If x is the measured value of a variable and Δx is the measurement error, then:

Estimated propagated error: $dy = f'(x)dx \approx f'(x)\Delta x$

Estimated relative error: $\dfrac{dy}{y}$

Example 5: The measurement of the radius of a circle is 20 inches with a maximum error of 0.10 inch. Approximate the maximum propagated error and the relative error in computing the area and the circumference of the circle.

Area:

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r\, dr$

Propogated Error in Area $= dA\Big|_{\substack{r=20 \\ dr=0.10}} = 2\pi \overset{in}{(20)} \overset{in}{(0.10)} = \boxed{4\pi \text{ in}^2}$

Relative error $= \frac{dA}{A} = \frac{4\pi \text{ in}^2}{\pi (20 \text{in})^2} = \frac{4\pi \text{ in}^2}{400\pi \text{ in}^2} = 0.01 \Rightarrow \boxed{1\% \text{ error}}$

Circumference

$C = 2\pi r$

$\frac{dC}{dr} = 2\pi$

$dC = 2\pi\, dr$

Propogated Error in Circumference $= dC\Big|_{\substack{dr=0.10 \\ r=20}} = 2\pi (0.10 \text{ in}) = \boxed{0.2\pi \text{ in}} \approx \boxed{0.628 \text{ in}}$

Relative Error $= \frac{dC}{C} = \frac{0.2\pi \text{ in}}{2\pi(20 \text{in})} = \frac{0.2\pi \text{ in}}{40\pi \text{ in}} = \frac{0.2}{40} = \frac{0.2}{40}\left(\frac{5}{5}\right) = \frac{1}{200} = \boxed{0.005}$

so $\boxed{0.5\% \text{ error}}$

Example 6: The measurements of the height and inside radius of a right cylinder are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propagated error and the relative error in computing the volume of the cylinder.

See Spring 2015 notes in Archives