5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

> Definition: $\log_b x = y$ is equivalent to $b^y = x$.
>
> The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
> b is called the *base* of the logarithm.

The logarithm of base e is called the *natural logarithm*, which is abbreviated "ln".

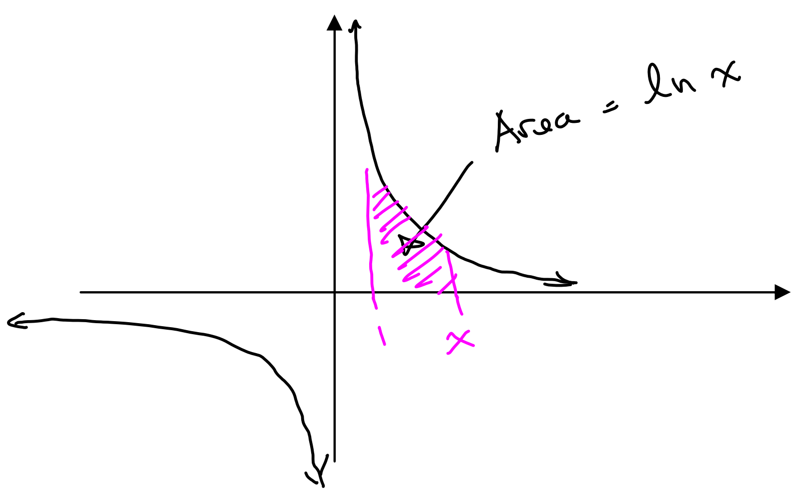
> The natural logarithm:
>
> $$\ln x = \log_e x.$$
>
> Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

A calculus approach to the natural logarithm:

> The natural logarithm function is defined as
>
> $$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$



Consider the function $f(t) = \frac{1}{t}$

For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

Note: The integral is not defined for $x < 0$.

For $x = 1$, $\ln x = \int_1^1 \frac{1}{t}\,dt = 0$.

$0 < x < 1$

For $\cancel{x<}1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$.

Recall:

> The Fundamental Theorem of Calculus, Part II:
>
> Let f be continuous on the interval $[a,b]$. Then the function g defined by
>
> $$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b$$
>
> is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$.

$$\frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

$g(x) = \ln x = \int_1^x \frac{1}{t}\,dt$

$g'(x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$

This means that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

> The Derivative of the Natural Logarithmic Function
>
> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

★ Memorize!

Laws of Logarithms:

If x and y are positive numbers and r is a rational number, then:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

$\ln\left(\frac{1}{x}\right) = \ln(1) - \ln(x) = 0 - \ln(x)$

$= -\ln(x)$

3. $\ln\left(x^r\right) = r \ln x$

Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

$$\ln\left(\frac{x^3(x+5)^{1/2}}{x^2+4}\right) = \ln\left[x^3(x+5)^{1/2}\right] - \ln(x^2+4)$$
$$= \ln x^3 + \ln(x+5)^{1/2} - \ln(x^2+4)$$
$$= \boxed{3\ln x + \frac{1}{2}\ln(x+5) - \ln(x^2+4)}$$

Ex 1½

Note:

$$\ln\left(\frac{x}{(x^2+6)(x-5)}\right)$$
$$\ln(x) - \ln\left[(x^2+6)(x-5)\right]$$
$$= \ln(x) - \left[\ln(x^2+6) + \ln(x-5)\right]$$
$$= \boxed{\ln(x) - \ln(x^2+6) - \ln(x-5)}$$

The graph of $y = \ln x$**:**

It can be shown that $\lim_{x\to\infty} \ln x = \infty$ and that $\lim_{x\to 0^+} = -\infty$.

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

For $x > 0$, $\frac{dy}{dx} = \frac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0, \infty)$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

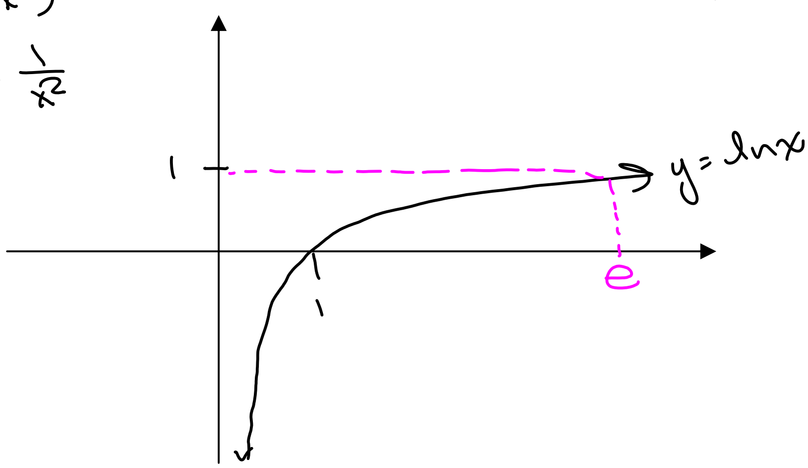
Note: $\ln x$ is only defined for $x > 0$

For $x > 0$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0, \infty)$.

2nd deriv.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$$

$\ln(1) = 0$



Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values $\left(\lim_{x\to\infty} \ln x = \infty\right)$, the Intermediate Value Theorem guarantees that there is a number *x* such that $\ln x = 1$. That number is called *e*.

$$e \approx 2.71828182845904523536$$

(*e* is in irrational number—it cannot be written as a decimal that ends or repeats.)

Recall: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

$$\frac{dy}{dx} = \frac{1}{2x^5+3x} \frac{d}{dx}(2x^5+3x)$$

$$= \frac{1}{2x^5+3x}(10x^4+3)$$

$$= \boxed{\frac{10x^4+3}{2x^5+3x}}$$

Ex: $y = (2x^5+3x)^3$

$$\frac{dy}{dx} = 3(2x^5+3x)^2 \frac{d}{dx}(2x^5+3x)$$

$$= \boxed{3(2x^5+3x)^2(10x^4+3)}$$

Ex: $y = \sin(2x^5+3x)$

$$\frac{dy}{dx} = \cos(2x^5+3x)\frac{d}{dx}(2x^5+3x)$$

$$= \boxed{\cos(2x^5+3x)(10x^4+3)}$$

Note: $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or, written another way, $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}(\ln(\cos x))$.

$$\frac{d}{dx}(\ln(\cos x)) = \frac{1}{\cos x}\frac{d}{dx}(\cos x)$$

$$= \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = \boxed{-\tan x}$$

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

$$f(x) = (\ln x)^{-1}$$

$$f'(x) = -1(\ln x)^{-2}\frac{d}{dx}(\ln x) = -1(\ln x)^{-2}\left(\frac{1}{x}\right)$$

$$= \boxed{-\frac{1}{x(\ln x)^2}}$$

Example 5: Find the derivative of $f(x) = x^2 \ln x$.

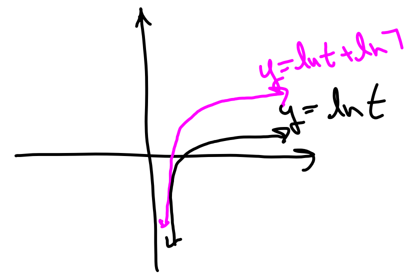
$$f'(x) = x^2\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x^2)$$

$$= x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = \boxed{x + 2x\ln x}$$

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

<u>Example 7:</u> Find the derivative of $g(t)=\ln(7t)$.

$$g'(t) = \frac{1}{7t}\cdot\frac{d}{dt}(7t) = \frac{1}{7t}\cdot 7 = \boxed{\frac{1}{t}}$$



<u>Note</u>:

$$g(t) = \ln(7t) = \ln(7) + \ln(t)$$

$$g'(t) = 0 + \frac{1}{t}$$

$$= \frac{1}{t}$$

<u>Example 8:</u> Determine the derivative of $f(x)=\dfrac{\ln 6x}{(x+4)^5}$.

Logarithmic differentiation:

To differentiate $y=f(x)$:

1. Take the natural logarithm of both sides.

2. Use the laws of logarithms to expand.

3. Differentiate implicitly with respect to *x*.

4. Solve for $\dfrac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y = (x^2+2)^5(2x+1)^3(6x-1)^2.$$

$$\ln y = \ln\left[(x^2+2)^5(2x+1)^3(6x-1)^2\right]$$

$$\ln y = 5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 5\left(\frac{1}{x^2+2}\right)(2x) + 3\left(\frac{1}{2x+1}\right)(2) + 2\left(\frac{1}{6x-1}\right)(6)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}$$

solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right]$$

$$\boxed{\frac{dy}{dx} = (x^2+2)^5(2x+1)^3(6x-1)^2\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right]}$$

Example 10: Find y' for $y = \dfrac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}$.