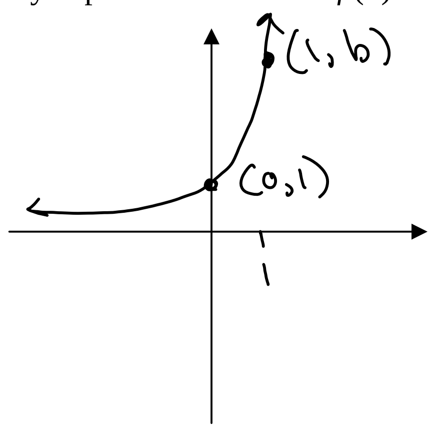
5.4: Exponential Functions: Differentiation and Integration

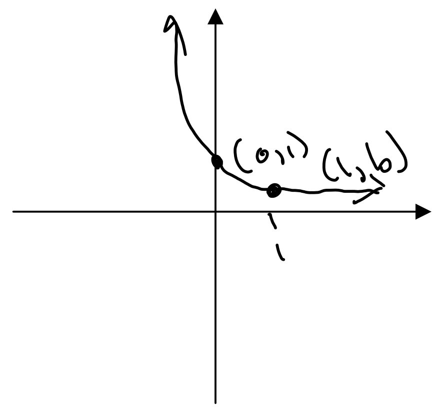
Short Review:

An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

For any exponential function $f(x) = b^x$, the graph looks like one of the following.



$b > 1$



$0 < b < 1$

Notice:
- Domain is $(-\infty, \infty)$.
- Range is $(0, \infty)$.
- Horizontal asymptote is $y = 0$.
- Always passes through the points (0,1) and (1,b).

The natural exponential function:

The number *e* can be defined in several ways.

> One definition of the number *e*:
>
> *e* is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

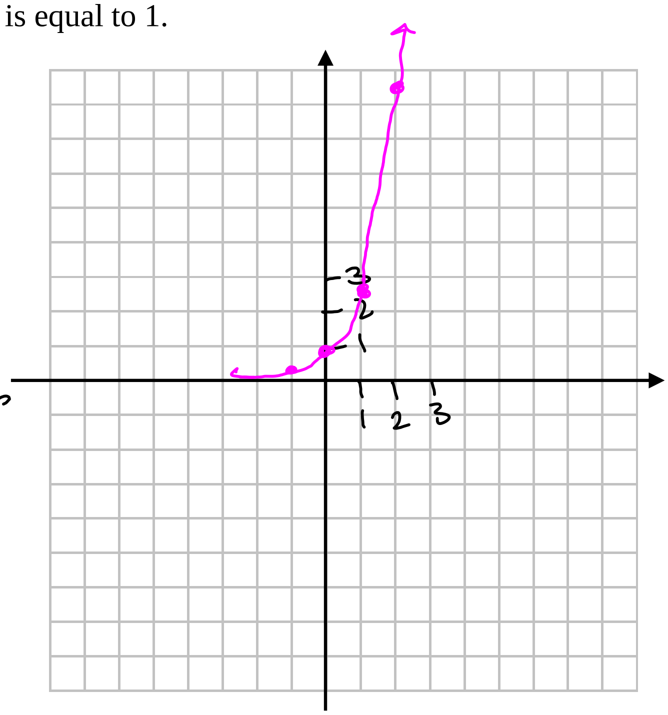
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$\boxed{e \approx 2.718281828459}$$

The slope of the tangent line at the point $(0,1)$ is equal to 1.

The graph of $f(x) = e^x$**:**

x	$y = e^x$
-1	$e^{-1} = \frac{1}{e} \approx \frac{1}{3}$
0	$e^0 = 1$
1	$e^1 = e \approx 2.7$
2	$e^2 \approx$ a litt less than 9



Another definition of the number *e*:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \text{or, equivalently,} \quad e = \lim_{x\to 0}(1+x)^{1/x}$$

☆ know it

$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$e = \lim_{x\to 0}\left(1+x\right)^{1/x}$

Derivatives of exponential functions:

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

In otherwords, if $f(x) = e^x$, then $f'(x) = e^x$.

Start with $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

and let $h = \frac{1}{x}$.

Then $hx = 1$

$x = \frac{1}{h}$

as $x \to \infty$, $h = \frac{1}{x} \to 0$

This gives us

$e = \lim_{h\to 0}\left(1+h\right)^{1/h}$

Example 1: Find the derivative of $f(x) = -7e^x$.

$\frac{d}{dx}\left(-7e^x\right) = -7\frac{d}{dx}\left(e^x\right)$

$= \boxed{-7e^x}$

Example 2: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

$f(x) = 5(e^x+7)^{1/2}$

$f'(x) = 5\left(\frac{1}{2}\right)(e^x + 7)^{-1/2} \frac{d}{dx}(e^x+7)$

$= \frac{5}{2}(e^x+7)^{-1/2}(e^x+0)$

$= \frac{5e^x}{2}(e^x+7)^{-1/2} = \boxed{\frac{5e^x}{2\sqrt{e^x+7}}}$

Example 3: Find the derivative of $f(x) = e^x \sin x$.

Use product rule.

Example 4: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

$g'(x) = e^{-7x}\frac{d}{dx}(-7x) + 6x^2 + 0 = e^{-7x}(-7) + 6x^2$

$= \boxed{-7e^{-7x} + 6x^2}$

Example 5: Find the derivative of $y = e^{x^2+4x}$.

$\frac{dy}{dx} = e^{x^2+4x}\frac{d}{dx}(x^2+4x) = \boxed{e^{x^2+4x}(2x+4)}$

Example 6: Find the derivative of $f(x) = \cos(e^x - x)$.

Example 7: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0,9)$.

Integration of exponential functions:

$$\boxed{\int e^x dx = e^x + c}$$

<u>Example 8:</u> Determine $\int (x^2 - 5e^x)\, dx$

$$\int x^2 dx - 5\int e^x dx$$

$$= \boxed{\frac{x^3}{3} - 5e^x + c}$$

<u>Example 9:</u> Find $\int e^{5t} dt$.

$$\int \underbrace{e^{5t}}_{e^u} \underbrace{dt}_{\frac{1}{5}du} = \frac{1}{5}\int e^u du$$

$$= \frac{1}{5}e^u + c$$

$$= \boxed{\frac{1}{5}e^{5t} + c}$$

$u = 5t$

$\frac{du}{dt} = 5$

$du = 5\,dt$

$\frac{1}{5}du = dt$

Check:

$$\frac{d}{dt}\left(\frac{1}{5}e^{5t}\right)$$

$$= \frac{1}{5}e^{5t}\frac{d}{dt}(5t)$$

$$= \frac{1}{\cancel{5}}e^{5t}(\cancel{5})$$

$$= e^{5t} \checkmark$$

<u>Example 10:</u> Find $\int_1^3 e^{2x-3} dx$.

$$\int_1^3 e^{2x-3} dx$$

$$= \frac{1}{2}\int_{u=-1}^{u=3} e^u du$$

$$= \frac{1}{2}e^u \Big|_{u=-1}^{u=3}$$

$$= \frac{1}{2}e^3 - \frac{1}{2}e^{-1} = \boxed{\frac{1}{2}\left(e^3 - \frac{1}{e}\right)}$$

$u = 2x-3$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}du = dx$

$x = 1 \Rightarrow u = 2(1) - 3 = -1$

$x = 3 \Rightarrow u = 2(3) - 3 = 3$

Example 11: Find $\int te^{t^2}\,dt$.

Example 12: Determine $\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx$.

Example 13: Determine $\int \frac{e^x - e^{-x}}{e^{3x}}\,dx$