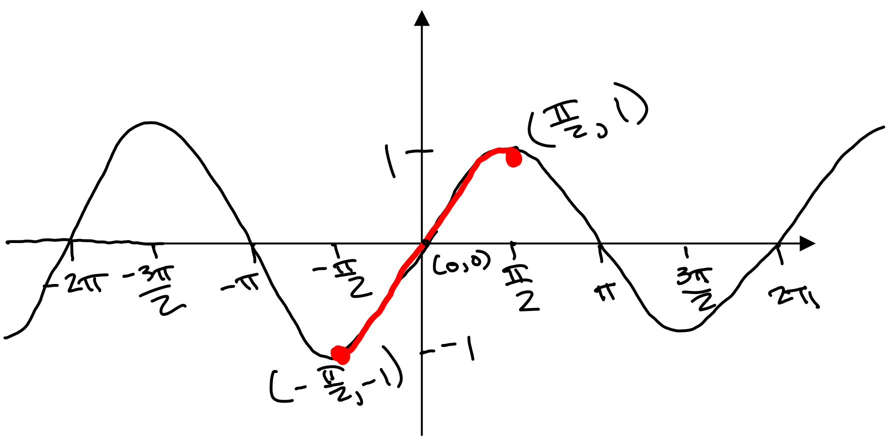
5.6: Inverse Trigonometric Functions – Differentiation

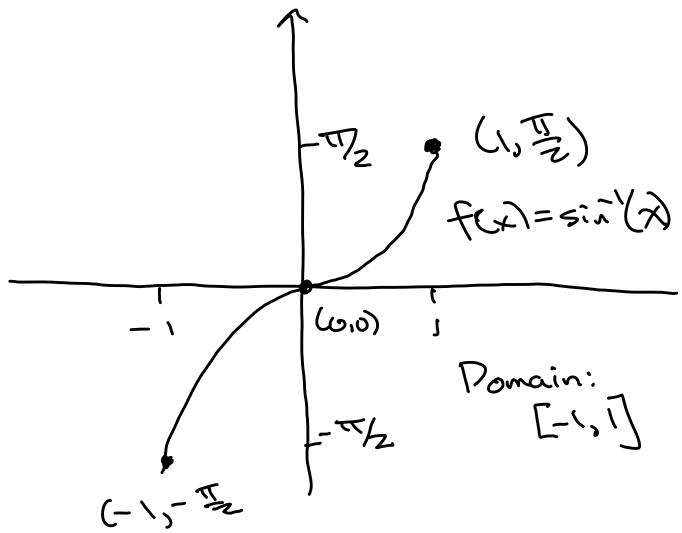
Because none of the trigonometric functions are one-to-one, none of them have an inverse function. To overcome this problem, the domain of each function is restricted so as to produce a one-to-one function.

y = sin x fails the horizontal line test and so does not have an inverse.

> **Inverse sine function:**
>
> $$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \ \text{ and } \ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



we restrict the domain of y = sin x to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Domain: $[-1, 1]$

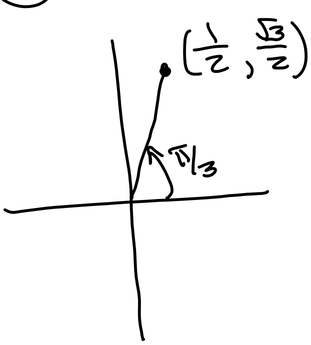
Properties of the inverse sine function:

$$\sin^{-1}(\sin x) = x \ \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \ \text{ for } -1 \le x \le 1$$

Example 1: Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{2}\right)$.

(a) Let $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Then $\sin\theta = \frac{\sqrt{3}}{2}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



So $\theta = \frac{\pi}{3}$

$$\boxed{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}}$$

(b) Let $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$. Then $\sin\theta = -\frac{1}{2}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\theta = \boxed{\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}}$$

Ex: Find $\sin^{-1}\left(-\frac{\pi}{2}\right)$. If $\theta = \sin^{-1}(-\pi/2)$, then $\sin\theta = -\frac{\pi}{2} \approx -1.57$, and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note: $-\frac{\pi}{2} \approx -1.57$

impossible!!

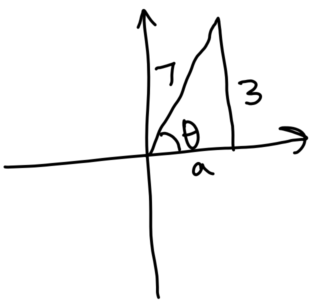
Always $-1 \le \sin\theta \le 1$

So $\boxed{\sin^{-1}\left(-\frac{\pi}{2}\right) \text{ is not defined.}}$

Example 2: Evaluate $\cot\left(\sin^{-1}\frac{3}{7}\right)$

Let $\theta = \sin^{-1}\left(\frac{3}{7}\right)$

Then $\sin\theta = \frac{3}{7}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



$a^2 + 3^2 = 7^2$

$a^2 = 40$

$a = \pm\sqrt{40} \Rightarrow$ choose $a = \sqrt{40}$

$\cot\left(\sin^{-1}\frac{3}{7}\right) = \cot\theta = \boxed{\frac{\sqrt{40}}{3}}$

$\frac{\text{adj}}{\text{opp}}$

Example 3: Evaluate $\sin\left(\sin^{-1}(-0.54)\right)$.

$\boxed{-0.54}$

Example 4: Evaluate $\sin\left(\sin^{-1} 2\right)$.

$\boxed{\text{Undefined}}$

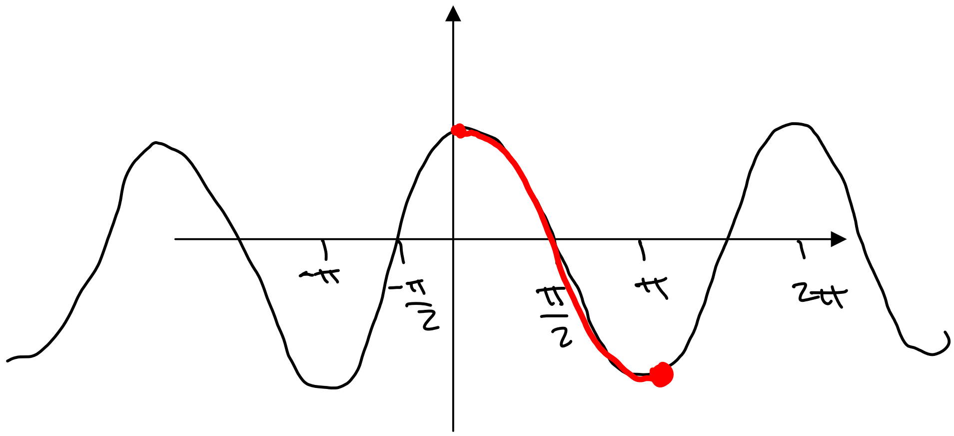
Example 5: Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$

Example 6: Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

Inverse cosine function:

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \ \text{ and } \ y \in [0, \pi]$$



Properties of the inverse cosine function:

$$\cos^{-1}(\cos x) = x \ \text{ for } \ 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \ \text{ for } \ -1 \le x \le 1$$

Example 7: Evaluate $\cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$

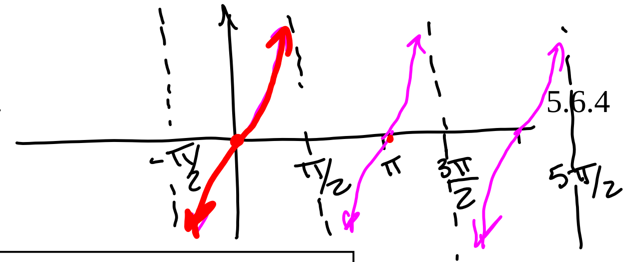
$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

then $\cos\theta = -\frac{1}{2}$ and $\theta \in [0, \pi]$

Example 8: Evaluate $\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right)$

$$\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$$

$y = \tan x$



Inverse tangent function:

Note:
arctan $x = \tan^{-1} x$
arcsin $x = \sin^{-1} x$
etc

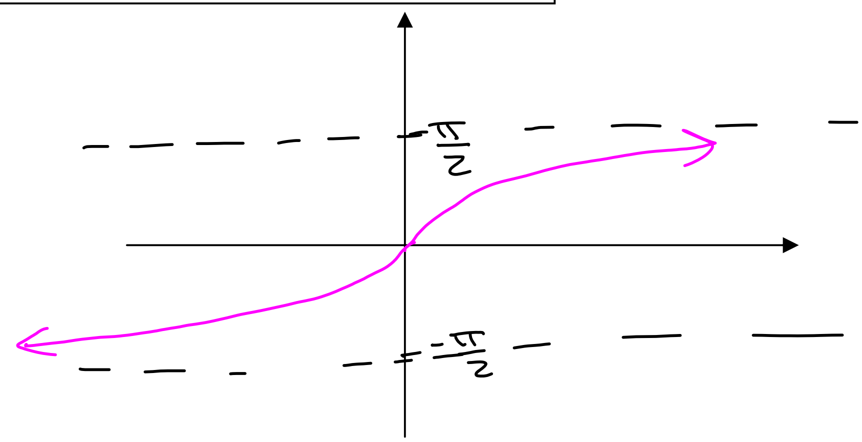
$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \ \text{and} \ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \ \text{and} \ \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Graph of $y = \tan^{-1} x$**:**

Know this graph!



Inverse secant, cosecant, and cotangent functions:

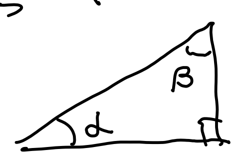
These are not used as often, and are not defined consistently. Our book defines them as follows:

$$y = \cot^{-1} x \quad \text{if and only if} \quad \cot y = x \ \text{and} \ y \in (0, \pi)$$

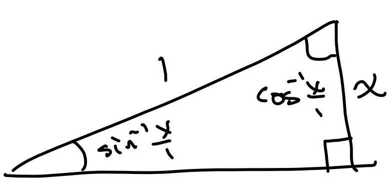
$$y = \csc^{-1} x \quad \text{if and only if} \quad \csc y = x \ \text{and} \ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \ \text{and} \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

Note: Angles in a triangle add up to π (180°)



$\beta + \alpha = \frac{\pi}{2}$ (90°)



An important identity:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

Differentiation of the inverse sine function:

Since $y = \sin x$ is continuous and differentiable, so is $y = \sin^{-1} x$.

We want to find its derivative.

$y = \sin^{-1} x$, Find $\frac{dy}{dx}$

Then $x = \sin y$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Implicit diff: $\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$

$1 = (\cos y)\frac{dy}{dx}$

$\frac{1}{\cos y} = \frac{dy}{dx}$

Need to write in terms of x: We know $\cos^2 y + \sin^2 y = 1$ (Pythag. trig identity)

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1 - \sin^2 y}$

We know $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos y > 0$. So $\cos y = \sqrt{1 - \sin^2 y}$

$\sin y = x \Rightarrow$

$\cos y = \sqrt{1 - x^2}$

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}} \quad , -1 < x < 1}$$

<u>Example 9:</u> Differentiate $f(x) = \sin^{-1}(2x - 7)$.

$$f'(x) = \frac{1}{\sqrt{1 - (2x-7)^2}} \cdot \frac{d}{dx}(2x-7)$$

$$= \frac{1}{\sqrt{1 - (2x-7)^2}}(2) = \boxed{\frac{2}{\sqrt{1 - (2x-7)^2}}}$$

Differentiation of the inverse cosine function:

$$\boxed{\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}} \quad ,-1<x<1}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$

Differentiation of the inverse ~~cosine~~ tangent function:

$$\boxed{\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}}$$

Derivatives of other inverse trigonometric functions:

Know these 3!

$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\csc^{-1}x\right)=-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$	$\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$

no need to memorize these 3

Example 10: Find the derivative of $f(x) = e^{\tan^{-1} 2x}$.

$f(x) = e^{\tan^{-1}(2x)}$

$f'(x) = e^{\tan^{-1}(2x)} \frac{d}{dx}\left(\tan^{-1}(2x)\right) = e^{\tan^{-1}(2x)} \cdot \frac{1}{1+(2x)^2} \cdot \frac{d}{dx}(2x)$

$= \boxed{e^{\tan^{-1}(2x)} \cdot \frac{2}{1+4x^2}}$

Example 11: Find the derivative of $y = \csc^{-1}(\tan x)$.

Example 12: Find the derivative of $f(x) = x^3 \arccos 2x$.

Example 13: Find the equation of the line tangent to the graph of $f(x) = \arctan x$ at the point where $x = -1$.