5.7: Inverse Trigonometric Functions – Integration

Two important integration rules come from the inverse trigonometric differentiation rules.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

Example 1: Find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$.

$$\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}(u) + c = \boxed{\frac{1}{3}\sin^{-1}(3x) + c}$$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

Example 2: Find $\int \frac{1}{x^2+7}\,dx$.

$$\int \frac{1}{x^2+7}\,dx = \int \frac{1}{7\left(\frac{x^2}{7}+1\right)}\,dx$$

$u = \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}x$

$\frac{du}{dx} = \frac{1}{\sqrt{7}}$

$du = \frac{1}{\sqrt{7}}\,dx$

$\sqrt{7}\,du = dx$

$$= \frac{1}{7}\int \frac{1}{\left(\frac{x}{\sqrt{7}}\right)^2+1}\,dx$$

$$= \frac{1}{7}\cdot\sqrt{7}\int \frac{1}{u^2+1}\,du$$

$$= \frac{\sqrt{7}}{7}\tan^{-1}(u) + c = \boxed{\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + c}$$

More general forms of these integration rules are

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

Ex: $\int \frac{1}{x^2+5}\,dx$

$= \int \frac{1}{x^2+(\sqrt{5})^2}\,dx$

$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

use formula to left with $a=\sqrt{5}$

Example 3: Find $\int \frac{4x}{x^4+25}\,dx$.

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$\int \frac{4x}{x^4+25}\,dx = \int \frac{4x}{(x^2)^2+25}\,dx$$

$$= 4\int \frac{x}{(x^2)^2+25}\,dx = 4\left(\frac{1}{2}\right)\int \frac{1}{u^2+25}\,du$$

$$= 2\int \frac{1}{u^2+5^2}\,du = 2\left(\frac{1}{5}\right)\arctan\left(\frac{u}{5}\right)+C$$

(use above formula with $a=5$)

$$= \boxed{\frac{2}{5}\arctan\left(\frac{x^2}{5}\right)+C}$$

Another antiderivative:

$$\boxed{\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x|+C}$$

Example 4: Find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

$u = \frac{x}{2} = \frac{1}{2}x$
$du = \frac{1}{2}dx$
$2\,du = dx$
$2u = x$

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}}\,dx$$

$$= \int \frac{1}{x\sqrt{4}\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx = \frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx$$

$$= \frac{1}{2}\cdot 2\int \frac{1}{x\sqrt{u^2-1}}\,du = \frac{1}{\cancel{2}}\cdot\cancel{2}\int \frac{1}{2u\sqrt{u^2-1}}\,du$$

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$$= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{1}{2}\sec^{-1}|u| + c$$

$$= \boxed{\frac{1}{2}\sec^{-1}\left|\frac{x}{2}\right| + c}$$

Example 5: Find $\int \frac{1}{4x^2-12x+17}\,dx$.

If possible, make it look like $\underbrace{\qquad\qquad\qquad}_{\int \frac{1}{u^2+a^2}\,du}$

$$\int \frac{1}{4x^2-12x+17}\,dx$$

$$= \int \frac{1}{4\left(x-\frac{3}{2}\right)^2+8}\,dx = \frac{1}{4}\int \frac{1}{\left(x-\frac{3}{2}\right)^2+2}\,dx$$

$u = x - \frac{3}{2}$
$du = dx$

$$= \frac{1}{4}\int \frac{1}{u^2+2}\,du = \frac{1}{4}\int \frac{1}{u^2+(\sqrt{2})^2}\,du$$

use $a = \sqrt{2}$

$$= \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \boxed{\frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{2}}\right) + C}$$

Complete the square:

$$4x^2 - 12x + 17$$
$$= (4x^2 - 12x) + 17$$
$$= 4(x^2 - 3x) + 17$$
$$= 4\left(x^2 - 3x + \frac{9}{4}\right) + 17 - \cancel{4}\left(\frac{9}{\cancel{4}}\right)$$

$$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

$$= 4\left(x - \frac{3}{2}\right)^2 + 8$$

Example 6: Find $\int \frac{x-7}{\sqrt{5-4x^2}}\,dx$.

$$\int \frac{x-7}{\sqrt{5-4x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{5-4x^2}}\,dx - \int \frac{7}{\sqrt{5-4x^2}}\,dx$$