

## 5.8: Hyperbolic Functions

The unit circle  $x^2 + y^2 = 1$  is used in trigonometry to define the trigonometric functions. Often they are called “circular functions.” In a similar manner, the unit hyperbola  $x^2 - y^2 = 1$  can be used to define functions that have many similar characteristics as the circular trigonometric functions. Such functions are called “hyperbolic functions.” These hyperbolic functions can also be defined in terms of the natural exponential function.

sinh: hyperbolic sine  
 cosh: hyperbolic cosine  
 etc.

$\sinh x = \frac{e^x - e^{-x}}{2}$	$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\operatorname{coth} x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Recall: odd and even functions

We can also think of decomposing  $e^x$  into odd and even parts:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

The odd part is  $\frac{e^x - e^{-x}}{2} = \sinh x$  and the even part is  $\frac{e^x + e^{-x}}{2} = \cosh x$ .

**Example 1:** Evaluate (a)  $\cosh 2$  (b)  $\tanh 0$  (c)  $\sinh 1$

$$\textcircled{a} \quad \cosh 2 = \frac{e^2 + e^{-2}}{2}$$

$$\textcircled{b} \quad \tanh (0) = \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

Hyperbolic functions have many uses in science and engineering applications.

### Hyperbolic identities:

Several identities involving hyperbolic functions are similar to the usual trigonometric identities, but there are a few important differences.

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh 2t = 2 \sinh t \cosh t$	
$\cosh 2t = \cosh^2 t + \sinh^2 t$	

### Derivatives of hyperbolic functions:

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Note:

$$\begin{aligned} \frac{d}{dx}(\sinh bx) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) \\ &= \frac{1}{2}(e^x - e^{-x}(-1)) \\ &= \frac{1}{2}(e^x + e^{-x}) \\ &= \cosh bx \quad \checkmark \end{aligned}$$

### Example 2:

Find  $\frac{dy}{dx}$  for  $y = \tanh \sqrt{3x^2 - 5}$ .

$$\begin{aligned} \frac{dy}{dx} &= \operatorname{sech}^2(3x^2 - 5)^{\frac{1}{2}} \frac{d}{dx}(3x^2 - 5)^{\frac{1}{2}} \\ &= \left[ \operatorname{sech}^2(3x^2 - 5)^{\frac{1}{2}} \right] \left( \frac{1}{2} (3x^2 - 5)^{-\frac{1}{2}} \right) (6x) \\ &= \frac{3x}{\sqrt{3x^2 - 5}} \operatorname{sech}^2(3x^2 - 5)^{\frac{1}{2}} = \frac{3x}{\sqrt{3x^2 - 5}} \cdot \left( \frac{2}{e^{\sqrt{3x^2 - 5}} + e^{-\sqrt{3x^2 - 5}}} \right)^2 \end{aligned}$$

### Inverse hyperbolic functions:

The functions  $\sinh$  and  $\tanh$  are one-to-one functions, which means they have inverse functions. The function  $\cosh$ , however, is not one-to-one. But, restricting the domain of  $\cosh$  to  $[0, \infty)$  defines a one-to-one function.

$$y = \sinh^{-1} x \text{ if and only if } \sinh y = x$$

$$y = \cosh^{-1} x \text{ if and only if } \cosh y = x \text{ and } y \geq 0$$

$$y = \tanh^{-1} x \text{ if and only if } \tanh y = x$$

Switch variables

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

Solve for y:

$$2x = e^y - e^{-y}$$

∴ etc

Because the hyperbolic functions are defined in terms of the natural exponential function, it should come as no surprise that the inverse hyperbolic functions are defined in terms of the natural logarithmic function.

*no need to memorize*

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), x \in \mathbb{R}$
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$
$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

$\mathbb{R}$  = set of real numbers

### Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{ x \sqrt{x^2+1}}$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\frac{d}{dx}(\operatorname{coth}^{-1} x) = \frac{1}{1-x^2}$

**Note:** Even though the derivative of  $\tanh^{-1} x$  and  $\operatorname{coth}^{-1} x$  look identical, they are really different because these inverse functions are defined on different domains:  $|x| < 1$  for  $\tanh^{-1} x$ , and  $|x| > 1$  for  $\operatorname{coth}^{-1} x$ .

**Example 3:** Find  $\frac{d}{dx}[\sinh^{-1}(\tan x)]$ .

$$\frac{d}{dx}[\sinh^{-1}(\tan x)] = \frac{1}{\sqrt{1+\tan^2 x}} \cdot \sec^2 x$$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|}$$

$$= \begin{cases} \sec x & \text{if } \sec x > 0 \\ -\sec x & \text{if } \sec x < 0 \end{cases}$$

$$= |\sec x| \quad \text{as long as } \sec x \neq 0$$

Recall:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

**Example 4:**  $\int \frac{3}{\sqrt{x^2-1}} dx$

**Example 5:** Evaluate  $\int \frac{4dx}{\sqrt{9x^2-1}}$ .

Use  $\frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

$$\begin{aligned} & 4 \int \frac{1}{\sqrt{(3x)^2-1}} dx \\ &= 4 \left(\frac{1}{3}\right) \int \frac{1}{\sqrt{u^2-1}} du \\ &= \frac{4}{3} \cosh^{-1} u + c = \boxed{\frac{4}{3} \cosh^{-1}(3x) + c} \end{aligned}$$

$$\begin{aligned} u &= 3x \\ \frac{du}{dx} &= 3 \\ du &= 3 dx \\ \frac{1}{3} du &= dx \end{aligned}$$

**Example 6:** Evaluate  $\int \coth x dx$

$$\int \coth x dx = \int \frac{\cosh x}{\sinh x} dx = \int \cosh x \left( \frac{1}{\sinh x} \right) dx$$

$$\begin{aligned} u &= \sinh x \\ \frac{du}{dx} &= \cosh x \\ du &= \cosh x dx \end{aligned}$$

$$= \int \frac{1}{u} du$$

$$\begin{aligned} &= \ln|u| + c = \ln|\sinh x| + c \\ &= \ln \left| \frac{e^x - e^{-x}}{2} \right| + c \end{aligned}$$

$$= \ln|e^x - e^{-x}| - \ln|2| + c$$

*throw altogether into c*

$$= \boxed{\ln|e^x - e^{-x}| + \hat{c}}$$