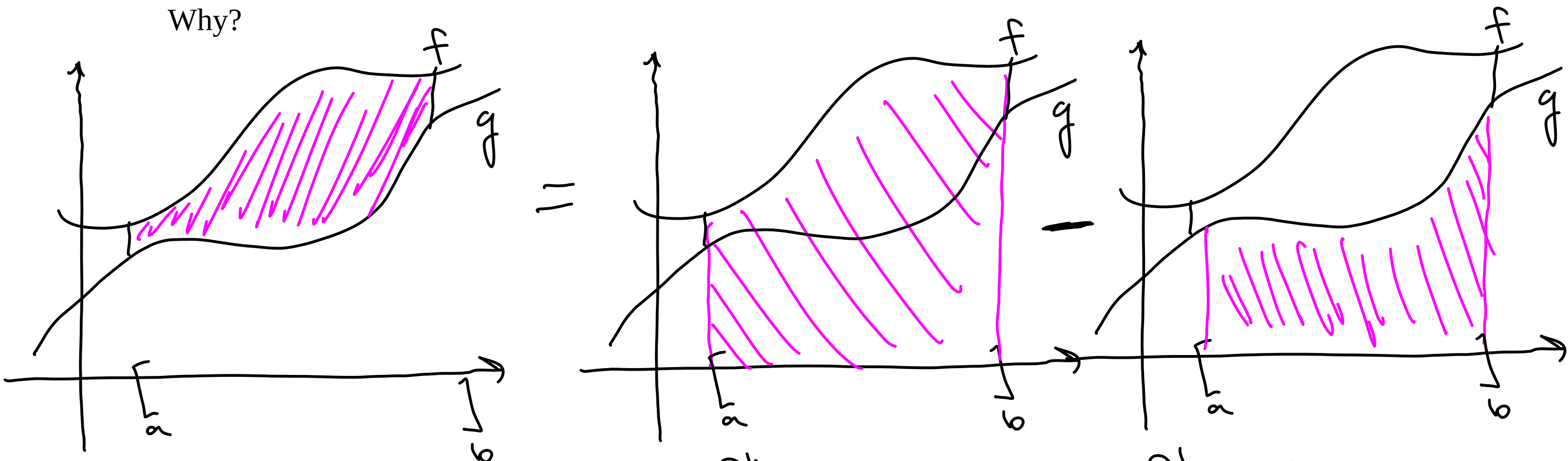
7.1: Area of a Region Between Two Curves

Because the definite integral represents the “net” area under a curve, we can use integration to find the area between curves.

If f and g are continuous and $f(x) \geq g(x)$ on $[a,b]$, then the area between $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b [f(x) - g(x)]dx$$

Why?

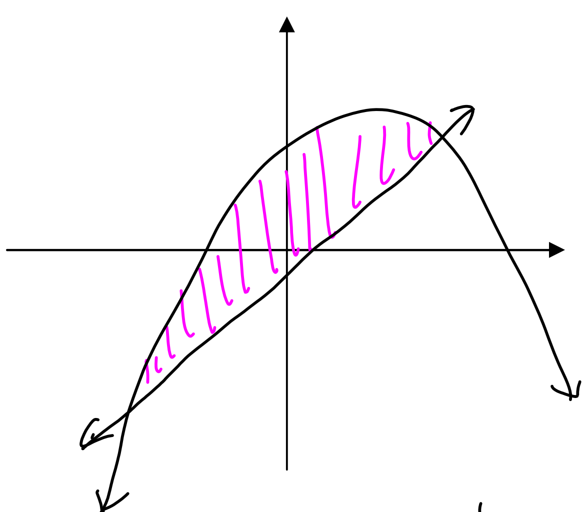


Area between f and g $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

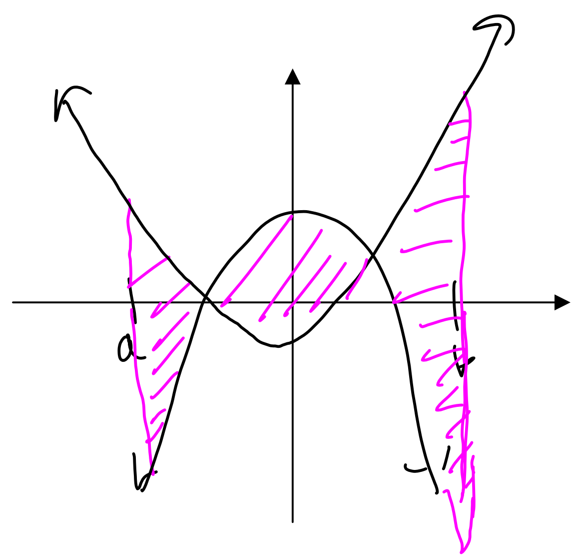
$= \int_a^b [f(x) - g(x)]dx$

Note: If $g(x) \geq f(x)$ on $[a,b]$, then $\int_a^b [f(x) - g(x)]dx$ is negative.

Some different types of area scenarios:

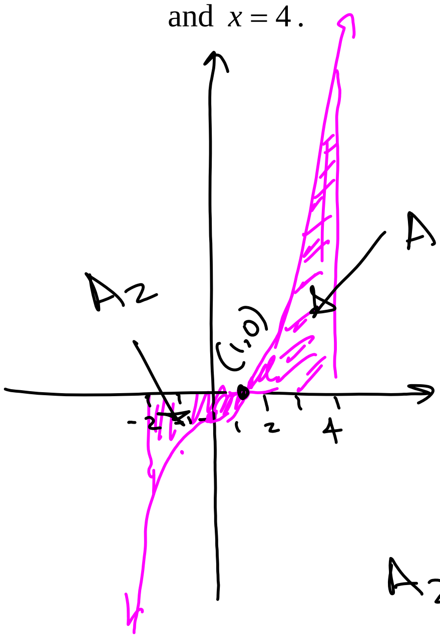


Find area enclosed by 2 curves



Area enclosed by 2 curves and 2 vertical lines

Example 1: Find the area of the region bounded by $f(x) = x^3 - 1$ and the lines $y = 0$, $x = -2$, and $x = 4$.



Find intersection pts: (x-intercepts)

$y = x^3 - 1$ and $y = 0$

Set y's equal: $x^3 - 1 = 0$

$x^3 = 1$

$x = \sqrt[3]{1} = 1$ so intersection pt is $(1,0)$

$$A_1 = \int_1^4 (x^3 - 1)\,dx = \left[\frac{x^4}{4} - x\right]\Bigg|_1^4 = \frac{4^4}{4} - 4 - \left[\frac{1^4}{4} - 1\right]$$

$= 64 - 4 - \left(\frac{1}{4} - 1\right)$

$= 60 - \left(-\frac{3}{4}\right) =$

$= 60 + \frac{3}{4} = \frac{243}{4}$

$= 60.75$

$$A_2 = \int_{-2}^{1} \left[0 - (x^3 - 1)\right]dx = -\int_{-2}^{1} (x^3 - 1)\,dx$$

OR let A_2 = signed area and then $A_2 = \int_{-2}^{1} (x^3 - 1)\,dx$

$= \left[\frac{x^4}{4} - x\right]\Big|_{-2}^{1} = \frac{1^4}{4} - 1 - \left[\frac{(-2)^4}{4} - (-2)\right]$

$= \frac{1}{4} - 1 - 4 - 2 = \frac{1}{4} - 7$

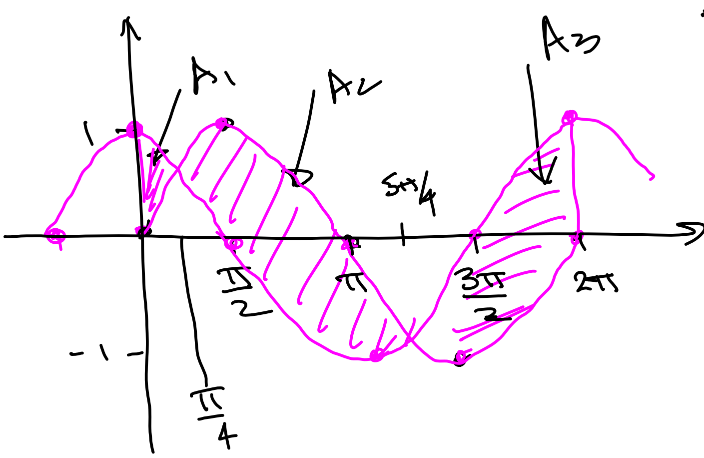
$= \frac{1}{4} - \frac{28}{4}$

$= -\frac{27}{4}$

$$\text{Area} = |A_1| + |A_2| = \left|\frac{243}{4}\right| + \left|\frac{-27}{4}\right| = \frac{243}{4} + \frac{27}{4} = \frac{270}{4} = \boxed{67.5}$$

Example 2: Find the area of the region bounded by $y = \cos x$, $y = \sin x$, and the lines $x = 0$, and $x = 2\pi$.

Let A_1, A_2, A_3 be the signed areas.



Find intersection points (really just the x-values)

Set y's equal: $\cos x = \sin x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$= \left[\sin x - (-\cos x)\right]\Big|_0^{\pi/4} = (\sin x + \cos x)\Big|_0^{\pi/4}$

$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \frac{2\sqrt{2}}{2} - 1$

$= \sqrt{2} - 1$

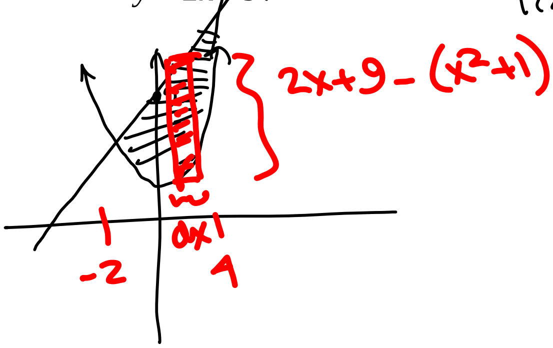
$$A_2 = \int_{\pi/4}^{5\pi/4} (\cos x - \sin x)\,dx = -2\sqrt{2}$$

$$A_3 = \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\,dx = 1 + \sqrt{2}$$

$$\text{Area} = |A_1| + |A_2| + |A_3| = |\sqrt{2} - 1| + |-2\sqrt{2}| + |1 + \sqrt{2}| \quad \boxed{4\sqrt{2}}$$

$= \sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2} =$

Example 3: Find the area of the region completely enclosed by the graphs of $y = x^2 + 1$ and $y = 2x + 9$.



Find intersection pts:

$$x^2 + 1 = 2x + 9$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4, \quad x = -2$$

$$\text{Area} = \int_{-2}^{4} \left[(2x+9) - (x^2+1) \right] dx$$

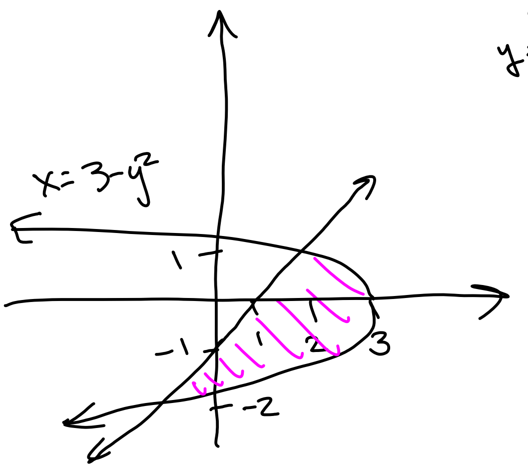
Example 4: Find the area of the region completely enclosed by the graphs of $y = x^3$ and $y = x$.

Example 5: Find the area of the region completely enclosed by the graphs of $x = y^2$ and $x = 4$.

Example 6: Find the area of the region completely enclosed by the graphs of $x = 3 - y^2$ and $x = y + 1$.

$x = y + 1$

$y = x - 1$



$$\int_{-2}^{1} \left[(3 - y^2) - (y+1) \right] dy$$

Find intersections: set x's equal:

$3 - y^2 = y + 1$

$0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$y = -2 \;,\; y = 1$