3.5: Limits at Infinity (cont'd)

Note Title 9/19/2016

Example 7: $h(x) = \dfrac{8x^2 - 6x + 1}{3x^2 + 4x - 5}$ Find the horizontal asymptote(s).

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\frac{8x^2}{x^2} - \frac{6x}{x^2} + \frac{1}{x^2}}{\frac{3x^2}{x^2} + \frac{4x}{x^2} - \frac{5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{8 - \frac{6}{x} + \frac{1}{x^2}}{3 + \frac{4}{x} - \frac{5}{x^2}} = \frac{\lim_{x\to\infty} 8 - \lim_{x\to\infty}\frac{6}{x} + \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{4}{x} - \lim_{x\to\infty}\frac{5}{x^2}}$$

$$= \frac{8 - 0 + 0}{3 + 0 - 0} = \frac{8}{3}$$

Recall:

$$\boxed{\text{Theorem: } \lim_{x\to\pm\infty} \frac{1}{x^r} = 0 \text{ for } r>0}$$

Similarly, $\lim_{x\to-\infty} h(x) = \dfrac{8}{3}$

Horizontal Asymptote: $\boxed{y = \frac{8}{3}}$

Ex 8: $\displaystyle \lim_{x\to\infty} \frac{7x^5 - 5x + 1}{8 - x^2} = \lim_{x\to\infty} \frac{\frac{7x^5}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{8}{x^2} - \frac{x^2}{x^2}}$

$$= \lim_{x\to\infty} \frac{7x^3 - \frac{5}{x} + \frac{1}{x^2}}{\frac{8}{x^2} - 1} = \frac{\lim_{x\to\infty} 7x^3 - 0 + 0}{0 - 1}$$

$$= \frac{\lim 7x^3}{-1} = \lim_{x\to\infty}\left(\frac{7x^3}{-1}\right) = \boxed{-\infty} \text{ (doesn't exist)}$$

Scratch

$$\frac{7(+\text{huge})^3}{-1}$$

$$\frac{+\text{huge}}{-1}$$

$$\Rightarrow -\text{huge}$$

$$\lim_{x\to-\infty} \frac{7x^5 - 5x + 1}{8 - x^2} = \lim_{x\to-\infty} \frac{\frac{7x^5}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{8}{x^2} - \frac{x^2}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{7x^3 - \frac{5}{x} + \frac{1}{x^2}}{\frac{8}{x^2} - 1} = \lim_{x\to-\infty} \frac{7x^3 - 0 + 0}{0 - 1}$$

next page

Ex 8 cont'd:

$$= \frac{\lim_{x\to-\infty}(7x^3)}{-1} = \boxed{\infty}$$ (does not exist)

Scratch: as $x \to -\infty$

$\frac{7(-\text{huge})^3}{-1} \Rightarrow \frac{-\text{huge}}{-1} \Rightarrow +\text{huge}$

(this means that $f(x) = \frac{7x^5 - 5x + 1}{8 - x^2}$ does not have a horizontal asymptote)

Ex 9: $$\lim_{x\to\infty} \frac{\sqrt[3]{x} - \sqrt{x}}{8 - x^{-2}} = \lim_{x\to\infty} \frac{\sqrt[3]{x} - \sqrt{x}}{8 - \frac{1}{x^2}}$$

Numerator: $\infty - \infty$ (indeterminate)

$$= \lim_{x\to\infty} \frac{x^{1/3} - x^{1/2}}{8 - \frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}\left(1 - x^{\frac{1}{2}-\frac{1}{3}}\right)}{8 - \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^{1/3}\left(1 - x^{1/6}\right)}{8 - \frac{1}{x^2}} = \boxed{-\infty}$$ (does not exist)

$\frac{(\text{huge})^{1/3}\left(1 - \text{huge}^{1/6}\right)}{8 - 0}$

$\frac{\sqrt[3]{\text{huge}}\left(1 - \sqrt[6]{\text{huge}}\right)}{8}$

$\Rightarrow \frac{+\text{huge}\,(-\text{huge})}{8}$

$\Rightarrow -\frac{\text{huge}}{8}$

$\Rightarrow -\text{huge}$

3.5 #45

$$\lim_{x\to-\infty}\left(x + \sqrt{x^2+3}\right)$$

$$= \lim_{x\to-\infty}\left(\frac{x + \sqrt{x^2+3}}{1}\right)\left(\frac{x - \sqrt{x^2+3}}{x - \sqrt{x^2+3}}\right)$$

$$= \lim_{x\to-\infty} \frac{x^2 - (x^2+3)}{x - \sqrt{x^2+3}}$$

Scratch

$-\text{huge} + \sqrt{(-\text{huge})^2 + 3}$

$\Rightarrow -\text{huge} + \sqrt{+\text{huge}}$

$\Rightarrow -\text{huge} + \text{huge}$

$\infty - \infty$ indeterminate form

see ~~next~~ page

#45 cont'd

$$= \lim_{x\to-\infty} \frac{-3}{x-\sqrt{x^2+3}} = 0$$

Scratch

As $x \to -\infty$,

$y \to \frac{-3}{-\text{huge} - \sqrt{(-\text{huge})^2+3}}$

$\Rightarrow \frac{-3}{-\text{huge} - \text{huge}}$

$\Rightarrow \frac{-3}{-\text{huge}}$

$\Rightarrow +$ tiny

Slant asymptotes:

A function has a slant (oblique) asymptote if the function values (y-values) approach those of a linear function $y=mx+b$ as x approaches ∞ or $-\infty$.

Ex: Does $f(x) = \frac{8x^2-7x+1}{2x-3}$ have a slant asymptote? If so, what is it?

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{8x^2}{x} - \frac{7x}{x} + \frac{1}{x}}{\frac{2x}{x} - \frac{3}{x}} = \lim_{x\to\infty} \frac{8x-7+\frac{1}{x}}{2-\frac{3}{x}}$$

$$= \frac{\left(\lim_{x\to\infty} 8x\right) - 7 + 0}{2-0} = \infty$$

Similarly, $\lim_{x\to-\infty} f(x) = -\infty$

If you end up with $\lim_{x\to\pm\infty} kx$ ~~to~~ 1st power, then there is a slant asymptote.

(For rational functions, there is a slant asymptote if the degree of the numerator is 1 more than the degree of the denominator. (i.e. if $\deg(\text{num}) - 1 = \deg(\text{denom})$))

To find the slant asymptote, use long division.

$$f(x) = \frac{8x^2 - 7x + 1}{2x - 3}$$

$$\frac{8x^2}{2x} \qquad \frac{5x}{2x}$$

$$\begin{array}{r} 4x + \frac{5}{2} \\ 2x-3 \enclose{longdiv}{8x^2 - 7x + 1} \\ -(8x^2 - 12x) \\ \hline +5x + 1 \\ -(5x - \frac{15}{2}) \\ \hline \frac{17}{2} \end{array}$$

Therefore,

$$f(x) = 4x + \frac{5}{2} + \frac{17/2}{2x-3}$$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(4x + \frac{5}{2}\right) + \lim_{x \to \pm\infty} \left(\frac{17/2}{2x-3}\right)$$

This term approaches 0 as $x \to \pm\infty$

As $x \to \pm\infty$, $f(x)$ becomes very similar to $y = 4x + \frac{5}{2}$

Slant asymptote: $\boxed{y = 4x + \frac{5}{2}}$

Ex: Use long division to decide whether

$$f(x) = \frac{2x^3 - x^2 + x}{x - 3}$$ has an asymptote.

$$\begin{array}{r} 2x^2 + 5x + 16 \\ x-3 \enclose{longdiv}{2x^3 - x^2 + x + 0} \\ -(2x^3 - 6x^2) \\ \hline 5x^2 + x \\ -(5x^2 - 15x) \\ \hline 16x + 0 \\ -(16x - 48) \\ \hline 48 \end{array}$$

So $f(x) = 2x^2 + 5x + 16 + \frac{48}{x-3}$

$$\lim_{x \to \pm\infty} \frac{48}{x-3} = 0,$$

So $f(x)$ is getting closer and closer to $y = 2x^2 + 5x + 16$ as $x \to \pm\infty$.

f approaches the parabola $y = 2x^2 + 5x + 16$ asymptotically.