10.3: Inference for Two Population Means: Variances Not Assumed Equal

If the variable x is normally distributed in both populations, then $\bar{x}_1 - \bar{x}_2$ is also normally distributed. Therefore, the standardized version of $\bar{x}_1 - \bar{x}_2$,

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$ is normally distributed.

However, in practice, we usually cannot know the population standard deviations σ_1 and σ_2. We must estimate them from the sample standard deviations. If we replace σ_1 and σ_2 in the above formula with s_1 and s_2, then the resulting test statistic is generally <u>not</u> normally distributed—instead, it follows the t-distribution.

<u>Distribution of the Unpooled t-Statistic (the Welsh's t)</u>

If x is normally distributed in each of two populations (or if both sample sizes are at least 30), then the test statistic

usually 0 because of $H_0: \mu_1 = \mu_2$

same as $\mu_1 - \mu_2 = 0$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1 - \bar{x}_2}} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

Δ: delta

approximately follows a t-distribution with Δ degrees of freedom, where

$$\Delta = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \dfrac{\left(\dfrac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$, rounded <u>down</u> to the nearest integer.

The equation for the degrees of freedom, Δ, is known as the Welsh-Satterthwaite equation.

Usually, we're testing the hypothesis that $\mu_1 = \mu_2$, which is equivalent to $\mu_1 - \mu_2 = 0$. In this case, our test statistic t becomes

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1 - \bar{x}_2}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}.$$

When comparing the means of two samples, we assume each sample was taken from a different population. In other words, Sample 1 comes from a population with mean μ_1; Sample 2 comes from a population with mean μ_2.

In order to use the procedure below, the following statements should be true:

- We know (or can reasonably assume) that the both populations follow the normal distribution, or have a large sample size ($n \geq 30$).
- Both samples are randomly obtained from their corresponding populations, or individuals are assigned randomly to one of the two groups.
- Observations within each sample are independent of one another.
- For each sample, the sample size is not over 5% of the population.
- The two samples are independent (data points from one sample are not paired with data points in the other sample).

<u>Hypothesis Testing for a the Difference of Two Independent Means:</u>

<u>Step 1</u>: Determine the significance level α.

<u>Step 2</u>: Determine the null and alternative hypotheses.

<u>Step 3</u>: Using your α level and hypotheses, sketch the rejection region.

<u>Step 4</u>: Compute the test statistic $t = \dfrac{\overline{x}_1 - \overline{x}_2}{s_{\overline{X}_1 - \overline{X}_2}} = \dfrac{\overline{x}_1 - \overline{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

find critical value

<u>Step 5</u>: Compute the degrees of freedom

$$\Delta = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

<u>Step 6</u>: Use a table (Table IV, on page A-6 and A-7) to determine the <u>critical value for *t*</u> associated with your rejection region.

<u>Step 7</u>: Determine whether the value of *t* calculated from your sample is in the rejection region.

- If *t* is in the rejection region, reject the null hypothesis.
- If *t* is not in the rejection region, do not reject the null hypothesis.

<u>Step 8</u>: State your conclusion.

Constructing a confidence interval for the difference between means:

We can construct a confidence interval for the difference between means for two independent samples.

$$\text{Upper bound: } (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2} \cdot s_{\bar{X}_1 - \bar{X}_2} = (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$\text{Lower bound: } (\bar{x}_1 - \bar{x}_2) - t_{\alpha/2} \cdot s_{\bar{X}_1 - \bar{X}_2} = (\bar{x}_1 - \bar{x}_2) - t_{\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

To calculate $t_{\alpha/2}$, we use the Student's *t*-table, with degrees of freedom Δ:

$$\Delta = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}.$$

<u>Note</u>: For null hypothesis $H_0 : \mu_1 = \mu_2$, you can use the confidence interval to test the hypothesis:

If the confidence interval for your α contains 0, then <u>do not reject</u> H_0.
If the confidence interval for your α does not contains 0, then <u>reject</u> H_0.

→ Assume the times for mold are normally distributed.

Example 1: Suppose an inventor wants to test the effectiveness of a new produce container intended to extend the life of refrigerated fruit. The inventor buys three dozen apples, and randomly chooses 12 of them to test in the new containers. (Due to limited funding for creation of prototypes, she only had 12 containers.) The other 24 apples were placed in standard bags. The containers and the bags were placed in the refrigerator. She recorded the time it took for each apple to develop observable mold. For the apples in the new containers, the mean time to develop mold was 6.5 days, with a standard deviation of 0.9 days. For the apples in bags, the mean time to develop mold was 5.2 days, with a standard deviation of 1.2 days. Perform a hypothesis test to evaluate whether the new containers helped the apples last longer, using $\alpha = 0.05$. Construct the 95% confidence interval for the difference between the means.

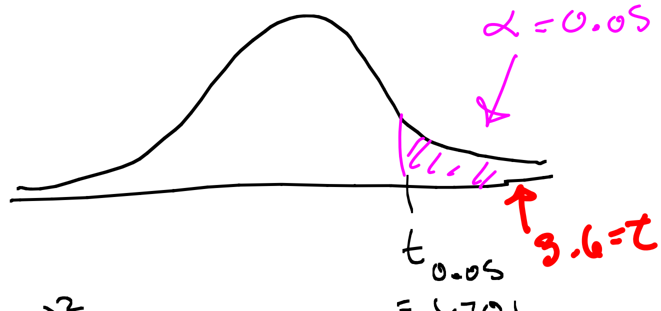
(new containers)
Sample 1
$n_1 = 12$
$\bar{x}_1 = 6.5$
$s_1 = 0.9$

(standard bags)
Sample 2
$n_2 = 24$
$\bar{x}_2 = 5.2$
$s_2 = 1.2$

$H_0: \mu_1 = \mu_2$
$H_a: \mu_1 > \mu_2$
$\alpha = 0.05$

$$A = \frac{s_1^2}{n_1} = \frac{(0.9)^2}{12} = 0.0675$$

$$B = \frac{s_2^2}{n_2} = \frac{(1.2)^2}{24} = 0.06$$



$$\Delta = df = \frac{(A+B)^2}{\frac{A^2}{11} + \frac{B^2}{23}} = \frac{(0.0675 + 0.06)^2}{\frac{(0.0675)^2}{11} + \frac{(0.06)^2}{23}} = 28.483 \Rightarrow \text{round } \underline{\text{down}} \text{ to } 28$$

(always round the df $\underline{\text{down}}$. If $\Delta = 28.79$, then use $df = 28$)

From t-table, for $df = 28$, the (and area = 0.05) critical value is $t_{0.05} = 1.701$

Find t for our sample:

standard error: $\sigma_{\bar{x}_1 - \bar{x}_2} \approx \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{A+B} = \sqrt{0.0675 + 0.06}$
$= 0.3571$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{6.5 - 5.2}{0.3571} = 3.64$$

This t is in rejection region, so we $\boxed{\text{Reject } H_0}$.

This sample provides evidence the new containers are more effective.

Both samples have $n \geq 30$, so OK to use t-test.

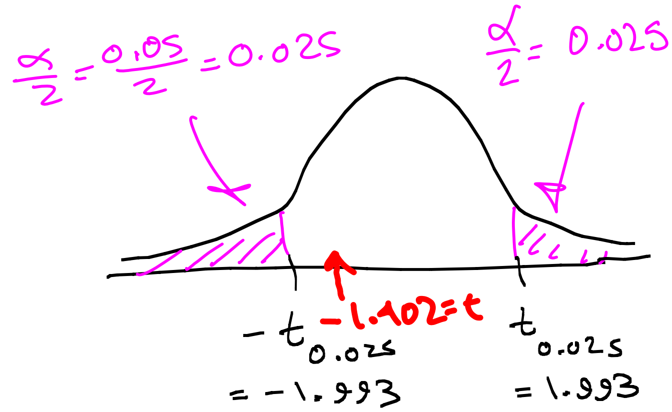
Example 2: The instructor for a keyboarding classes wishes to determine which of two sets of practice exercises is more effective for his students learning to type. One sample of 34 students used Exercise Set A and had a mean of 48.6 words per minute and a standard deviation of 6.2 words per minute on a typing test. A second sample of 42 students used Exercise Set B and had a mean of 50.9 words per minute and a standard deviation of 8.1 words per minute. Does this sample provide evidence that the two exercise sets resulted in different typing speeds? Use $\alpha = 0.05$. Construct the 95% confidence interval.

A = Pop 1
B = Pop 2

Sample A info
$n_1 = 34$
$\bar{x}_1 = 48.6$
$s_1 = 6.2$

Sample B info
$n_2 = 42$
$\bar{x}_2 = 50.9$
$s_2 = 8.1$

$H_0: \mu_1 = \mu_2$
$H_a: \mu_1 \neq \mu_2$
$\alpha = 0.05$



Find degrees of freedom:

$$df: \Delta = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$

$$= \frac{\left(\frac{(6.2)^2}{34} + \frac{(8.1)^2}{42}\right)^2}{\frac{\left(\frac{(6.2)^2}{34}\right)^2}{33} + \frac{\left(\frac{(8.1)^2}{42}\right)^2}{41}} = \frac{7.2508}{0.09825} = 73.79686$$

Round down to $df = 73$

Using t-table, for $df = 73$, area $= 0.025$, we see that the critical value is $t_{0.025} = 1.993$

Calculate t for our sample:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sigma_{\bar{x}_1 - \bar{x}_2}}$$

standard error: $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

$$\sigma_{\bar{x}_1 - \bar{x}_2} \approx \sqrt{\frac{(6.2)^2}{34} + \frac{(8.1)^2}{42}}$$

$$= \sqrt{2.69273} \approx 1.64095$$

$$= \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$\approx \frac{48.6 - 50.9}{1.64095} = \frac{-2.3}{1.64095} \approx -1.402.$$

$\boxed{\text{Do not reject } H_0}$

This sample does not provide evidence that the two sets of practice exercise differ in their effectiveness

Find 95% Confidence Interval

Upper bound: $\bar{x}_1 - \bar{x}_2 + t_{0.025}\, \sigma_{\bar{x}_1 - \bar{x}_2}$

$= 48.6 - 50.9 + 1.993(1.64095)$

$= 0.9704 \approx 0.97$

Lower bound: $48.6 - 50.9 - 1.993(1.64095)$

$= -5.57$

95% CI: $(-5.57, 0.97)$

Easiest approach for calculating degrees of freedom, etc in this problem:

Let $A = \frac{s_1^2}{n_1} = \frac{(6.2)^2}{34} = 1.130588$. Store this in calculator, under variable A.

Let $B = \frac{s_2^2}{n_2} = \frac{(8.1)^2}{42} = 1.562141$. Store this in calculator under Variable B.

Then our df formula becomes:

$$df = \Delta = \frac{(A+B)^2}{\frac{A^2}{33} + \frac{B^2}{41}} \approx \frac{7.2508}{0.09825} = 73.7969$$

Use the A's and B's in std. error also: $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{A+B}$

≈ 1.64095