4.3: Some Rules of Probability

Example 1: Need-based financial aid for college students can take the form of grants (do not need to be repaid) or loans (must be repaid). Consider a group of 70 students in which 30 students received grants, 35 received loans, and 13 received both. How many of these students received need-based financial aid?

G = students getting grants, L = students getting loans.

We want to find $n(G \cup L)$

$n(G) = 30$

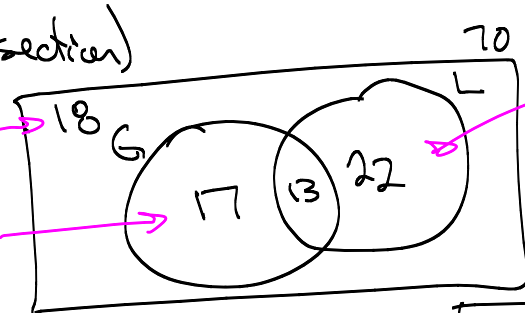
$n(L) = 35$

$n(G \cap L) = 13$ (get both ... intersection)

35-13 = 22

70-17-13-22 = 18

30-13 = 17



Number of students receiving aid is

$17 + 13 + 22 = \boxed{52}$

Notation: $n(A)$ means the number of elements in set A.

OR $n(G \cup L) = n(G) + n(L) - n(G \cap L) = 30 + 35 - 13 = 65 - 13 = \boxed{52}$

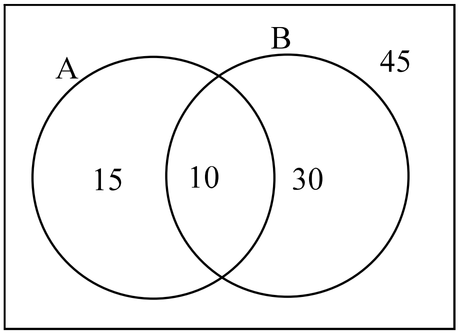
> Addition Principle for Counting
>
> For any two sets A and B,
>
> $$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$
>
> If A and B are mutually exclusive ($A \cap B = \varnothing$), then $n(A \cup B) = n(A) + n(B)$.
>
> *Mutually exclusive*: no outcomes in common (also called *disjoint events*).

Probability of unions and intersections:

> Probability of a Union of Two Events:
>
> $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
>
> If the two events are mutually exclusive (disjoint):
>
> $$P(A \cup B) = P(A) + P(B)$$

Example 2: Assume that an equally likely sample space is described by the Venn diagram below.

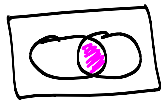
$n(U) = 15+10+30+45 = 100$



$P(A \cup B) = \frac{15}{100} + \frac{10}{100} + \frac{30}{100} = \frac{55}{100} = \boxed{0.55}$

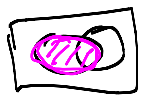
$P(A \cap B) = \frac{10}{100} = \boxed{0.10}$

equal



$P(B) = \frac{10}{100} + \frac{30}{100} = \frac{40}{100} = \boxed{0.40}$

$P(A) = \frac{15}{100} + \frac{10}{100} = \frac{25}{100} = \boxed{0.25}$

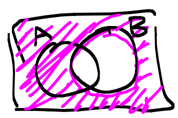


Note: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.40 - 0.10 = \boxed{0.55}$

Complements:

Probability of a complement:
$P(E^C) = 1 - P(E)$
$P(E) = 1 - P(E^C)$

$P(B') = \frac{15}{100} + \frac{45}{100} = \frac{60}{100} = \boxed{0.60}$



Example 1: Suppose that the probability of someone voting for a certain candidate is 0.46. What is the probability of not voting for the candidate?

E: vote for candidate

E^c: E-complement (not in E)

$P(E) = 0.46$

$P(E^c) = 1 - 0.46 = \boxed{0.54}$

Example 2: Consider the data below, from the Congressional Research Service.
https://fas.org/sgp/crs/misc/RS20811.pdf

Table 1. Distribution of Household Money Income by Selected Income Class, 2012

Income Class	Cumulative Freq	# of Households (in thousands)	% of Households
All Households		122,459	100.0
Less than $5,000	4204	4,204	3.4
$5,000 to $9,999	8933	4,729	3.9
$10,000 to $14,999	15915	6,982	5.7
$15,000 to $19,999	23072	7,157	5.8
$20,000 to $24,999		7,131	~~5.5~~ 5.8
$25,000 to $29,999		6,740	5.4
$30,000 to $34,999		6,354	5.2
$35,000 to $39,999		5,832	4.8
$40,000 to $44,999	54676	5,547	4.5
$45,000 to $49,999	59930	5,254	4.4
$50,000 to $59,999	69298	9,358	7.6
$60,000 to $69,999		8,305	6.8
$70,000 to $79,999		7,170	5.9
$80,000 to $89,999		5,969	4.9
$90,000 to $99,999		4,901	4.0
$100,000 to $124,999		9,490	7.7
$125,000 to $149,999		5,759	4.7
$150,000 to $199,999		6,116	5.0
$200,000 to $249,999		2,549	2.1
$250,000 and above		2,911	2.4
Median Income		$51,017	
Mean Income		$71,274	

Source: U.S. Census Bureau, 2012 Annual Social and Economic Supplement to the Current Population Survey.

$5K increments

$10K increments

$25K

$50K

lumped together

frequencies

error?

relative frequencies converted to percentages

(relative frequency is a decimal or fraction between 0 and 1)

b $< \$40K$

a $\geq \$100K$

2

Let X = household income

a) What is the probability that a randomly selected household has an income of $100,000 or more?

$$P(X \geq 100{,}000) = 0.077 + 0.047 + 0.05 + 0.021 + 0.024 = \boxed{0.219}$$

b) What is the probability that a randomly selected household has an income below $40,000?

$$P(X < \$40000) = 0.034 + 0.039 + 0.057 + 0.058 + 0.055 + 0.054 + 0.052 + 0.048 = \boxed{0.397}$$

c) What is the probability that a randomly selected household has an income ~~below $40,000?~~

d) What is the probability that a randomly selected household has an income below $250,000?

$P(D) = 1 - P(D^c) = 1 - P(X \geq 250\,000)$

$= 1 - 0.024 = \boxed{0.976}$

e) What is the probability that a randomly selected household has an income of $20,000 or more?

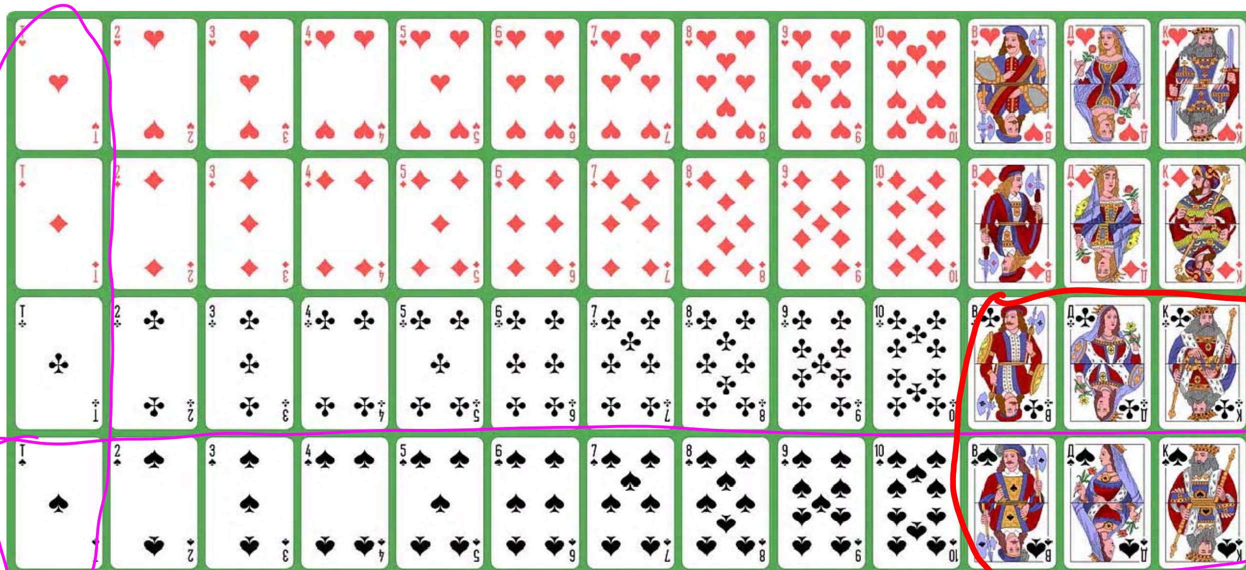
$E: X \geq \$20000$ | $P(E^c) = 0.188$

$E^c: X < \$20\,000$ | $P(E) = 1 - 0.188 = \boxed{0.812}$

f) Approximate the median household income.

half above M, half below, so need to find cutoff for $\frac{122459}{2} = 61229.5$. work out cumulative frequencies until you get to this

Example 3: Consider a standard deck of 52 cards.



a) What is the probability that a randomly selected card is a spade or a heart?

There are 13 spades, 13 heart, $Heart \cap Spades = \emptyset$

$P(Spade \cup Heart) = \frac{26}{52} = \boxed{\frac{1}{2}}$

b) What is the probability that a randomly selected card is a spade or an ace?

$P(Ace \cup Spade) = \frac{16}{52} = \boxed{\frac{4}{13}}$ OR $P(Ace \cup Spade)$

$= P(Ace) + P(Spade) - P(Ace \cap Spade)$

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

$= \boxed{\frac{4}{13}}$

c) What is the probability that a randomly selected card is not a black face card?

C = black face cards

There are 6 black face cards (in C)

so there are $52 - 6 = 46$ cards in C^c.

$P(C^c) = \frac{46}{52} = \boxed{\frac{23}{26}}$