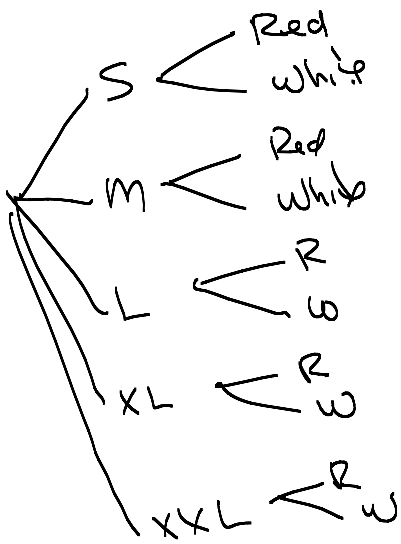
4.8: Counting Techniques (Permutations, Combinations, and the Multiplication Principle)

Multiplication principle for counting:

This principle is used to analyze sets which are determined by a sequence of operations.

Example 1: A company sells football jerseys. The jerseys come in sizes S, M, L, XL, and XXL. They also come in two colors: red for home games and white for away games. How many total types of jerseys does the company make?



There are 10 types of jerseys.

$5(2) = \boxed{10}$

Multiplication Principle

Suppose that n choices must be made, with

m_1 ways to make choice 1,
m_2 ways to make choice 2,
m_3 ways to make choice 3,
.
.
.
m_n ways to make choice n.

Then there are $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ ways to make the entire sequence of choices.

Example 2: How many license plate "numbers" can be formed by using a letter, followed by two digits, followed by three more letters?

$$\underset{\text{Letter}}{\underline{26}} \cdot \underset{\text{Digit}}{\underline{10}} \cdot \underset{\text{Digit}}{\underline{10}} \cdot \underset{\text{Letter}}{\underline{26}} \cdot \underset{\text{Letter}}{\underline{26}} \cdot \underset{\text{Letter}}{\underline{26}} = 26^4 \cdot 10^2 = \boxed{45697600}$$

How many can be formed assuming adjacent letters and numbers must be different?

$$\underline{26} \cdot \underline{10} \cdot \underline{9} \cdot \underline{26} \cdot \underline{25} \cdot \underline{25}$$

How many can be formed assuming letters and numbers cannot be repeated?

$$\underline{26} \cdot \underline{10} \cdot \underline{9} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23}$$

Products like this occur so frequently that special counting formulas and notations have been developed for them. These formulas use a function called the factorial.

> <u>The Factorial</u>:
>
> For a natural number (positive integer) n, $n!$ is called "n-factorial". It is defined as follows:
>
> $$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$
> $$n! = n(n-1)!$$
> $$0! = 1$$

$26! = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdots 3 \cdot 2 \cdot 1$

Example 3:

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{720}$

$$\frac{8!}{7!} = \frac{8 \cdot \cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \boxed{8}$$

$$\frac{97!}{3!94!} = \frac{97 \cdot 96 \cdot 95 \cdot 94 \cdot 93 \cdot 92 \cdots 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(94 \cdot 93 \cdot 92 \cdots)} = \frac{97 \cdot 96 \cdot 95 \cdot \cancel{94!}}{3 \cdot 2 \cdot 1 \cdot \cancel{94!}} = \boxed{147440}$$

Note: Factorials grow very rapidly!

Example 4: Compare 5!, 10!, and 15!.

$5! = 120$

$10! = 3628800$

$15! = 1.307674368 \times 10^{12}$

Permutations:

Example 5: Six horses are entered in a race. Assuming no ties, in how many possible ways can they finish first, second, and third?

$\underset{1st}{6} \cdot \underset{2nd}{5} \cdot \underset{3rd}{4} = \boxed{120}$ Note there are $6!$ possible finishing orders for Places #1 – 6.

This is a "permutation", or rearrangement.

Example 6: An Olympic event has $\boxed{\text{ten}}$ competitors. In how many ways can the gold, silver, and bronze medals be awarded (assuming no ties)?

$\underset{Gold}{10} \cdot \underset{Silver}{9} \cdot \underset{Bronze}{8} = 720$

here $n = 10$, $r = 3$

OR $P_{10,3} = \frac{10!}{(10-3)!} = \frac{10!}{7!}$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdots 1}{7 \cdot 6 \cdots 1}$

$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$

$= 10 \cdot 9 \cdot 8 = 720$

> Permutations of *n* Objects Taken *r* at a Time:
>
> The number of permutations of *n* objects taken *r* at a time without repetition is given by
>
> $$P_{n,r} = \frac{n!}{(n-r)!}$$
>
> Also $P_{n,r}$, $P(n,r)$, ${}_nP_r$, P_r^n
>
> Note: $P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$. So there are $n!$ ways to arrange *n* objects.

So for the last example, we have

Example: $P_{5,2} =$ $P_{5,2} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = \boxed{20}$

$P_{7,7} =$ $P_{7,7} = \boxed{5040}$ Note $7! = 5040$ also

$P_{4,3} =$ 24

Other notations: $P_{n,r}$ $\quad {}_nP_r$ $\quad P(n,r)$ $\quad P_r^n$

Combinations: $n=8,\ r=3$

Example 7: A student group with ~~five~~ eight officers must form a three-member committee. How many different committees can be formed?

$$\frac{n!}{(n-r)!\,r!} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{5!}\ \cancel{3\cdot 2\cdot 1}} = \boxed{56}$$

This is $C_{8,3}$, a combination

This is what we call a *combination* problem rather than a *permutation* problem.

Notice that in this situation, the order does not matter. In other words, "Joe, Mary, Sue" is the same committee as "Mary, Sue, Joe".

Combinations of *n* Objects Taken *r* at a Time:

The number of combinations of *n* objects taken *r* at a time without repetition is given by

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note: $C_{n,n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n!(1)} = 1.$

Note: $C_{8,3} = C_{8,5}$

$$\frac{8!}{5!\,3!} \qquad \frac{8!}{3!\,5!}$$

Back to our committees…

$$C_{8,3} = 56$$

Example 8: $C_{6,2} = \frac{6!}{4!\,2!} = 15$ Note $C_{6,4} = \frac{6!}{2!\,4!}$

$C_{7,4} =$ 35

$C_{10,9} = \boxed{10}$

$C_{2517,2516} = \boxed{2517}$

Other notations: ${}_nC_r$ $\quad C_{n,r}$ $\quad C(n,r)$ $\quad C_r^n$ $\quad \binom{n}{r}$

Example 9: A student group with five members must choose a president, vice-president, and treasurer. In how many ways can this be done?

$$\underbrace{5}_{\text{Pres}} \cdot \underbrace{4}_{\text{VP}} \cdot \underbrace{3}_{\text{Treasurer}} = P_{5,3} = \boxed{60}$$

Example 10: An art museum has a collection of 7 sculptures by a particular artist. There is only room to display four of the sculptures at a time. In how many different ways can four sculptures be chosen to display?

Different orders, are not considered different outcomes, so it's a combination

$$C_{7,4} = \boxed{35}$$

> **Very Important:**
> - If order matters, use permutations.
> - If order does not matter, use combinations.

Example 11: Consider a standard 52-card deck.

a. How many 5-card hands contain 5 hearts?

$$C_{13,5} = 1287$$

b. How many 5-card hands will contain exactly 2 aces and 2 queens?

52 − 4 aces − 4 queens = 44

$$\underbrace{C_{4,2}}_{\text{2 aces}} \cdot \underbrace{C_{4,2}}_{\text{2 queens}} \cdot \underbrace{44}_{\text{1 non-ace non-queen}} = 6 \cdot 6 \cdot 44 = \boxed{1584}$$

c. How many 5-card hands will contain 2 hearts and 3 clubs?

$$\underbrace{C_{13,2}}_{\text{2 hearts}} \cdot \underbrace{C_{13,3}}_{\text{3 clubs}} = 78 \cdot 286 = \boxed{22308}$$

Example 12: In how many ways can a quality-control engineer select a sample of 3 transistors for testing from a batch of 100 transistors?

$C_{100,3}$

Example 13: In how many different ways can a panel of 12 jurors and 2 alternate jurors be chosen from a group of 30 prospective jurors?

$$\underbrace{C_{30,12}}_{\text{12 jurors}} \cdot \underbrace{C_{18,2}}_{\text{2 alternates}} = 1.323 \times 10^{10}$$

Example 14: In how many ways can a subcommittee of four be chosen from a Senate committee of five Democrats and four Republicans if

a. All members are eligible?

$C_{9,4} = \boxed{126}$

b. The subcommittee must consist of two Republicans and two Democrats?

$$\underbrace{C_{5,2}}_{\text{2 republicans}} \cdot \underbrace{C_{4,2}}_{\text{2 Democrats}} = 10 \cdot 6 = \boxed{60}$$

Example 15: The members of a string quartet composed of two violinists, a violist, and a cellist are to be selected from a group of six violinists, three violists, and two cellists.

a. In how many ways can the string quartet be formed?

$$\underbrace{C_{6,2}}_{\text{2 violins}} \cdot \underbrace{3}_{\text{1 viola}} \cdot \underbrace{2}_{\text{1 cello}} = 15 \cdot 6 = \boxed{90}$$

b. In how many ways can the string quartet be formed if one of the violinists is to be designated as the first violinists and the other is to be designated as the second violinist?

$$\underbrace{6}_{\text{1st violin}} \cdot \underbrace{5}_{\text{2nd violin}} \cdot \underbrace{3}_{\text{viola}} \cdot \underbrace{2}_{\text{cellist}} = \boxed{180}$$

or $P_{6,2} \cdot 3 \cdot 2$

Example 16: A student planning her curriculum for the upcoming year must select one of five Business courses, one of three Mathematics courses, two of six elective courses, and either one of four History courses or one of three Social Science courses. How many different curricula are available for her consideration?

$$\underbrace{5}_{\text{Business}} \cdot \underbrace{3}_{\text{math}} \cdot \underbrace{C_{6,2}}_{\text{electives}} \cdot \underbrace{7}_{\text{Hist/ Social Science}}$$

$$= 5 \cdot 3 \cdot 15 \cdot 7$$

$$= \boxed{1575}$$

Example 17: Draw 5 cards from a standard 52-card deck.

a. What is the probability of getting 5 spades?

b. What is the probability of getting 2 kings, 2 queens, and a jack?

Sample space = set of all possible outcomes

S = {all possible 5-card hands}

$n(S) = C_{52,5} = 2\,598\,960$

A) # of ways to get 5 spades $= C_{13,5} = 1287$

A: 5 spades

$$P(A) = \frac{n(A)}{n(S)} = \frac{1287}{2\,598\,960} = 4.952 \times 10^{-4} \approx \boxed{0.0004952}$$

B): B: 2 Kings, 2 Queens, 1 Jack

$$n(B) = \underbrace{C_{4,2}}_{2K} \cdot \underbrace{C_{4,2}}_{2Q} \cdot \underbrace{4}_{1J} = 6 \cdot 6 \cdot 4 = 36 \cdot 4 = \boxed{144}$$

Example 18: Joe and Susan Thomas work in a department with 12 other people. Four employees are to be chosen to attend an important conference in the Bahamas. What is the probability that both Joe and Susan will be chosen?

S = all possible ways to choose 4 people to go

$n(S) = C_{14,4} = 1001$

E: Joe and Susan both chosen

$n(E) = \underbrace{1}_{\text{Joe}} \cdot \underbrace{1}_{\text{Susan}} \underbrace{C_{12,2}}_{\text{2 other people}} = 66$

$P(E) = \frac{n(E)}{n(S)} = \frac{66}{1001} \approx \boxed{0.0659}$

Example 19: Suppose that you had written six thank-you cards and addressed six envelopes. While you were away from your desk, your young child, attempting to be helpful, put the cards into the envelopes randomly, stamped and mailed them. What is the probability that all the cards were put into the correct envelopes?

Think of the envelopes as A, B, C, D, E, F

and the cards as a, b, c, d, e, f.

(Card a belongs in envelope A, etc)

How many ways can the cards get put in the envelopes?

$\underset{A}{\underline{6}} \cdot \underset{B}{\underline{5}} \cdot \underset{C}{\underline{4}} \cdot \underset{D}{\underline{3}} \cdot \underset{E}{\underline{2}} \cdot \underset{F}{\underline{1}} = 6! = 720$

So $n(S) = 720$

How many <u>correct</u> ways to put <u>all</u> the cards in <u>all</u> the envelopes: 1

So the probability of all being correct is $\boxed{\frac{1}{720}} \approx \boxed{0.0014}$