6.3: Working with Normally Distributed Variables

Recall: The *z*-score of a data point is its distance from the mean, measured in standard deviations.

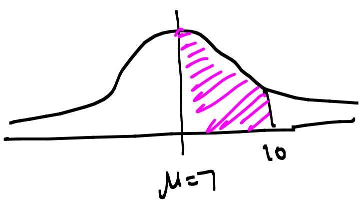
Recall: standard normal curve has $\mu = 0$, $\sigma = 1$

> Standardizing the values of a normal distribution:
>
> In a normal distribution with mean μ and standard deviation σ, where *x* is a data value, the *z*-score is
>
> $$z = \frac{x-\mu}{\sigma}.$$
>
> The area under a normal curve between $x = a$ and $x = b$ is the same as the area under the standard normal curve between the *z*-score for *a* and the *z*-score for *b*.

Example 1: Consider a normal curve with mean 7 and standard deviation 2.

a) What is the area under the curve between 7 and 10?

$\mu = 7$, $\sigma = 2$



Find z-score for $x = 10$:

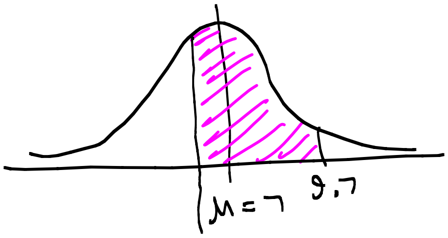
$z = \frac{x-\mu}{\sigma} = \frac{10-7}{2} = \frac{3}{2} = 1.50$

From table, area $= \boxed{0.4332}$ (look up $z = 1.50$

b) What is the probability that the variable is between 7 and 10?

$P(7 < X < 10) = P(0 < Z < 1.50) = \boxed{0.4332}$

c) What is the probability that the variable is between 6.5 and 9.7?



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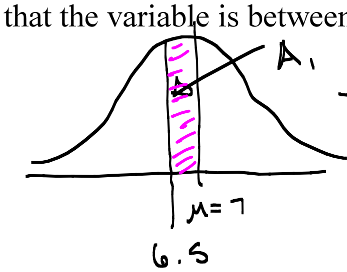

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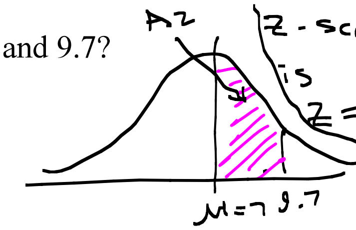


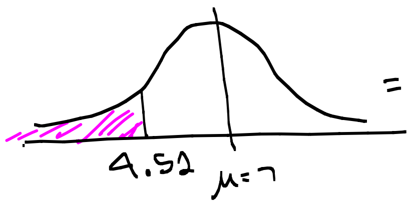
z-score for 6.5 is $z = \frac{x-\mu}{\sigma} = \frac{6.5-7}{2} = -0.25$

From Table, with $z = -0.25$, $A_1 = 0.0987$. From table, $z = 1.35$, $A_2 = 0.4115$

$P(6.5 < X < 9.7) = A_1 + A_2 = 0.0987 + 0.4115 = \boxed{0.5102}$

z-score for $x = 9.7$ is $z = \frac{x-\mu}{\sigma} = \frac{9.7-7}{2} = 1.35$

d) What is the probability that the variable is less than 4.52?



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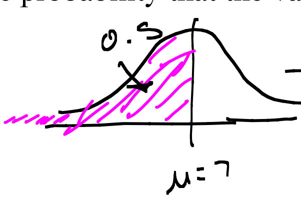

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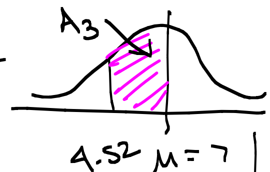


$z = \frac{x-\mu}{\sigma} = \frac{4.52-7}{2} = -1.24$

From table, $A_3 = 0.3925$

$P(X < 4.52) = 0.5 - A_3$

$= 0.5 - 0.3925 = \boxed{0.1075}$

<u>Properties of Normal Probability Distributions</u>:

1. $P(a \le x \le b) =$ area under the curve from a to b.
2. $P(-\infty \le x \le \infty) = 1 =$ total area under the curve.
3. $P(x = c) = 0$.

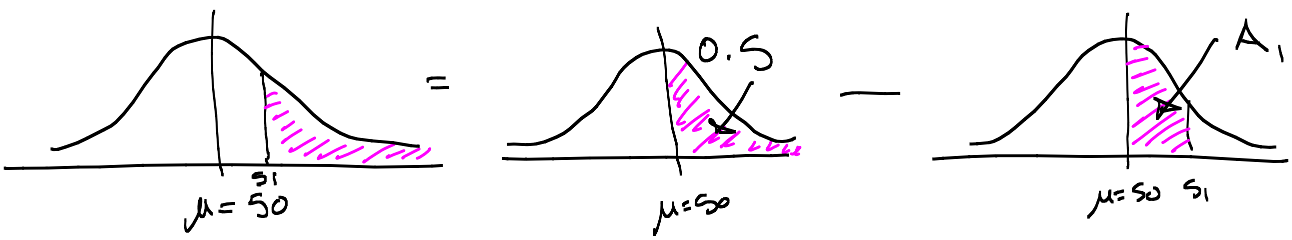
Note: $P(a \le x \le b) = P(a \le x < b) = P(a < x \le b) = P(a < x < b)$

<u>Example 2:</u> Dusty Dog Food Company ships dog food to its distributors in bags whose weights are normally distributed with a mean weight of 50 pounds and standard deviation 0.5 pound. If a bag of dog food is selected at random from a shipment, what is the probability that it weighs

a) More than 51 pounds?
b) Less than 49 pounds?
c) Between 49 and 51 pounds?
d) What is the percentage of dog food bags that weigh more than 51 pounds?

$\mu = 50, \quad \sigma = 0.5$

(a)



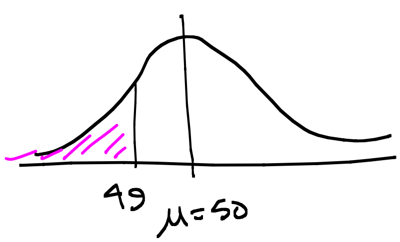
Find z-score for $x = 51$: $z = \frac{x-\mu}{\sigma} = \frac{51-50}{0.5} = 2.00$

From table, $A_1 = 0.4772$

$P(x>51) = 0.5 - A_1 = 0.5 - 0.4772 = 0.0228$

$\boxed{P(x>51) = 0.0228}$

(b)



Note: z-score for 49 is

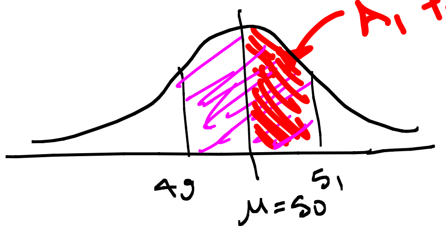
$z = \frac{x-\mu}{\sigma} = \frac{49-50}{0.5} = -2.00$

This is the mirror image of Part (a).

$\boxed{P(x<49) = 0.0228}$

From (a), $A_1 = 0.4772$

(c)



$P(49<x<51) = 2A_1 = 2(0.4772)$

$\boxed{P(49<x<51) = 0.9544}$

(d) From (a), $P(x>51) = 0.0228$.

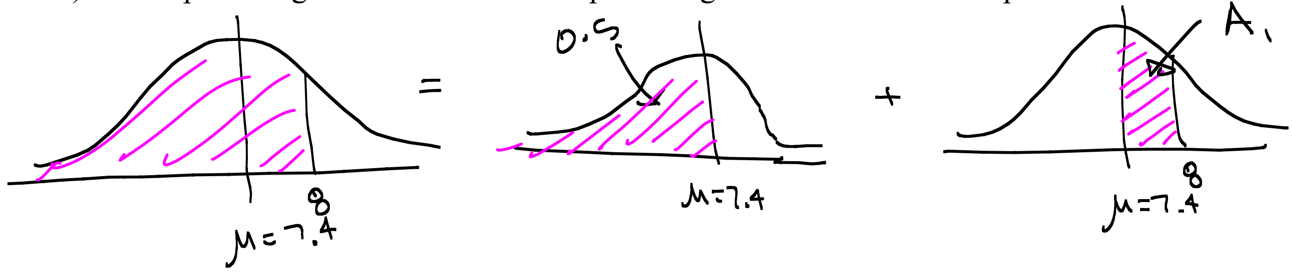
So $\boxed{2.28\% \text{ of bags weigh more than 51 lbs.}}$

Example 3: The medical records of infants delivered at a certain hospital show that the infants' birth weights in pounds are normally distributed with a mean of 7.4 and a standard deviation of 1.2.

$\mu = 7.4, \quad \sigma = 1.2$

a) What percentage of infants at this hospital weighed more than 9.2 pounds at birth?
b) What percentage of infants at this hospital weighed less than 8 pounds at birth?
c) What percentage of infants at this hospital weighed between 8 and 10 pounds at birth?

(b)



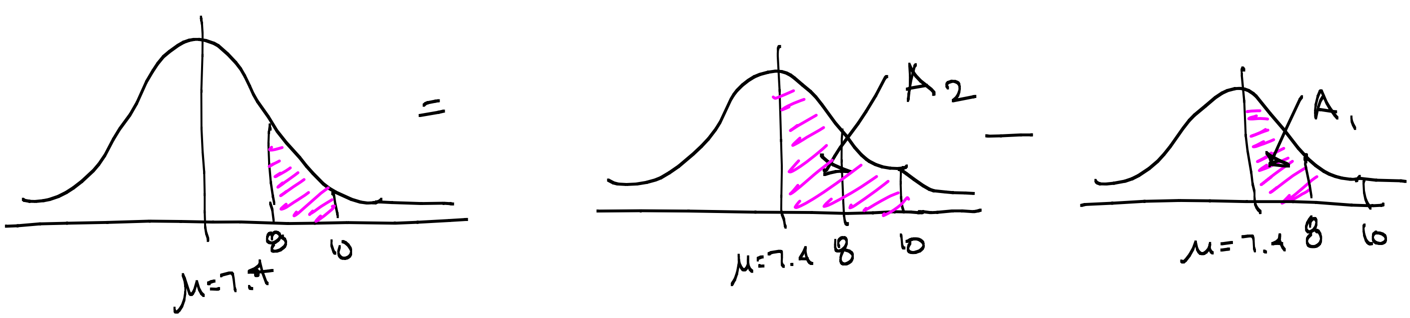
Find z-score for $x = 8$:

$$z = \frac{x - \mu}{\sigma} = \frac{8 - 7.4}{1.2} = 0.5.$$ From table, $A_1 = 0.1915$ (looking up $z = 0.50$).

$$P(X < 8) = 0.5 + A_1 = 0.5 + 0.1915 = 0.6915$$

So 69.15% of babies at this hospital weigh less than 8 lbs.

(c)



Find z-score for $x = 10$:

$$z = \frac{x - \mu}{\sigma} = \frac{10 - 7.4}{1.2} \approx 2.17.$$ Looking up $z = 2.17$ in table, $A_2 = 0.4850$

From Part (a), $A_1 = 0.1915$

$$P(8 < X < 10) = A_2 - A_1 = 0.4850 - 0.1915 = 0.2935 \Rightarrow$$

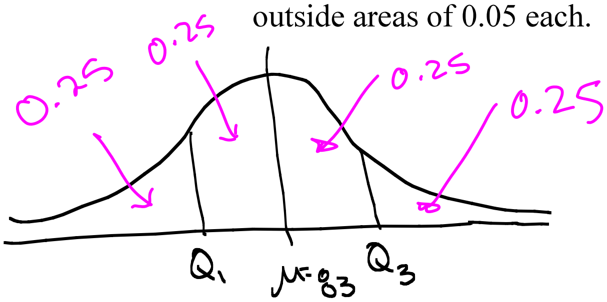
29.35% of babies are between 8 and 10 lbs

Important: The z-score is the number of standard deviations between the data point and the mean.

Example 4: A variable is normally distributed with mean 83 and standard deviation 24.

$\mu = 83, \quad \sigma = 24$

a) Find and interpret the quartiles.
b) Find and interpret the 98^{th} percentile.
c) Find and interpret the first and second deciles.
d) Find the value that 72% of all possible values of the variable exceed.
e) Find two values of the variable that divide the area into a middle area of 0.90 and two outside areas of 0.05 each.



Median
$= Q2$
$= \mu$

Note: $Q_3 = z_{0.25}$

Look up Area = 0.25 in table

It's between $A = 0.2486$ and $A = 0.2517$

So $z_{0.25} = Q_3 \approx 0.675$

Convert this z-score to an x.

$z = \frac{x - \mu}{\sigma}$

$z\sigma = x - \mu$

$z\sigma + \mu = x$

$x = 0.675(24) + 83$

$= 99.2$

$0.675 = \frac{x - 83}{24}$

$0.675(24) = x - 83$

$16.2 + 83 = x$

$99.2 = x$

From symmetry, for Q_1, we have $z = -0.675$

$z = \frac{x - \mu}{\sigma}$

$z\sigma = x - \mu$

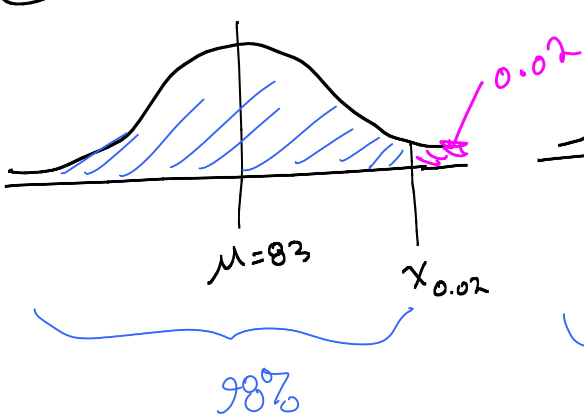
$x = z\sigma + \mu$

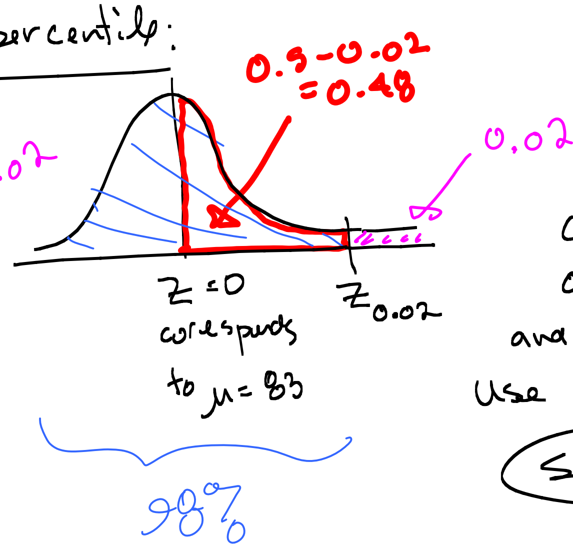
$x = -0.675(24) + 83$

$= 66.8$

$$\boxed{\begin{aligned} Q_1 &= 66.8 \\ Q_2 &= 83 \\ Q_3 &= 99.2 \end{aligned}}$$

(b) Find the 98th percentile:





Look up Area = 0.48 in Table.

Closest areas are 0.4798 (with $z = 2.05$) and 0.4803 (with $z = 2.06$)

Use $z = 2.055$

see next page

Example 4b cont'd:

b) we have $z = 2.055$

$$z = \frac{x-\mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$\mu + z\sigma = x$$

$$x = \mu + z\sigma$$

Given: $\mu = 83$, $\sigma = 24$

$x = 83 + 2.055(24) = \boxed{132.32}$ is the 98^{th} percentile.

(This means that about 98% of data points are below 132.32 and 2% are above).

c) Find 1^{st} and 2^{nd} deciles

Quartiles: break data set into 4 parts (same number of data pts in each part).

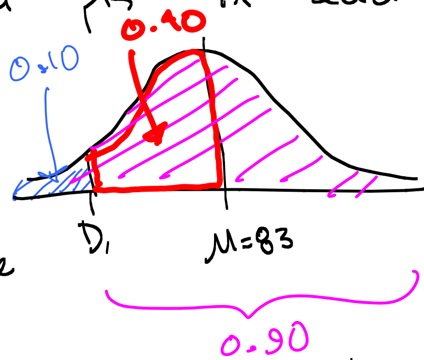
Deciles: Break data set into 10 parts (same # of data pts in each)

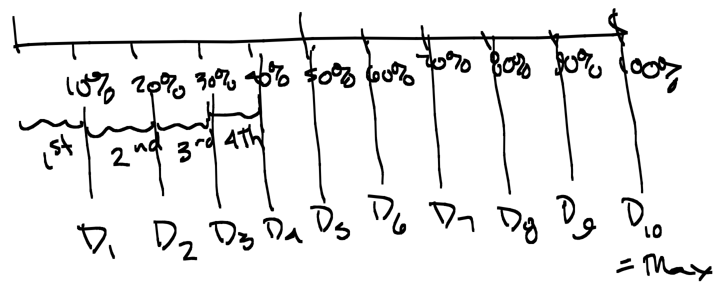
Find 1^{st} decile

1^{st} decile: D_1

10% are below

90% are above





Look up Area = 0.40 in table. Closest area is 0.3997, corresponding to $z = 1.28$.

D_1 is left of the mean, so use $z = -1.28$.

$$x = \mu + z\sigma$$

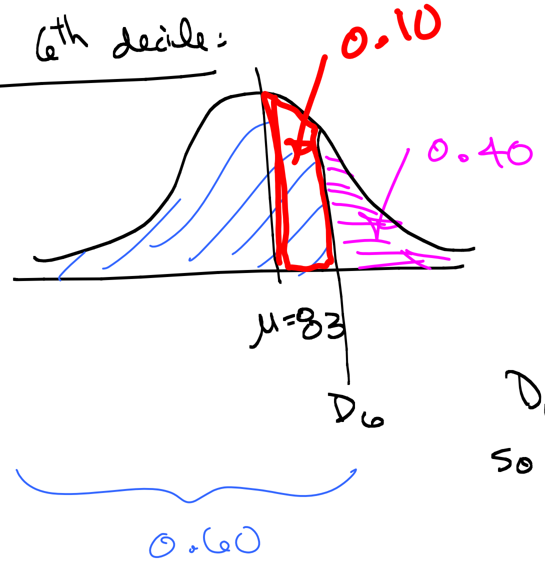
$$x = 83 - 1.28(24) = 52.28$$

$\boxed{1^{st} \text{ decile is } 52.28.}$

See next page

Ex 4 cont'd:

Find 6th decile:



For D_6: 60% of pts will be below D_6; 40% above

Look up Area = 0.10 in table:

Closest area is 0.0987, corresponding to $z = 0.25$

D_6 is to the right of the mean, so use the positive z-score.

$$x = \mu + z\sigma = 83 + 0.25(24) = 89$$

$\boxed{\text{6th decile is 89}}$

Example 5: The GPA of the senior class of a certain high school is normally distributed with a mean of 2.7 and a standard deviation of 0.4 point. If a senior in the top 10% of his or her class is eligible for admission to any state university, what is the minimum GPA that a senior should have to ensure eligibility to a state university?