

5.3: The Binomial Probability Distribution

Bernoulli trials:

A *Bernoulli trial* is an experiment with exactly two possible outcomes. We refer to one of the outcomes as a *success* (S) and to the other as a *failure* (F). We'll call the probability of success p and the probability of failure q . In other words,

$$P(S) = p \text{ and } P(F) = 1 - p = q. \text{ (Note that } S \text{ and } F \text{ are complements of one another.)}$$

Example 1: Roll a single die. Consider rolling a 5 to be a success.

Success: 1 roll a 5

Failure: 1 roll a 1, 2, 3, 4, 6

on 1 trial, $S = \{1, 2, 3, 4, 5, 6\}$
 $E = \{5\}$

$$p = P(\text{Success}) = \frac{1}{6}$$

$$q = P(\text{Failure}) = \frac{5}{6}$$

Example 2: Roll a pair of dice. Consider success to be rolling a sum of 7, 11, or 12.

On one trial: $S =$ set of 36 ordered pairs
 $n(S) = 36$

$$= \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ \vdots \\ (6,1), \dots, (6,6) \end{array} \right\}$$

$E = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (5,6), (6,5), (6,6)\}$

$$n(E) = 9$$

$$p = P(\text{Success}) = \frac{9}{36} = \frac{1}{4}, \quad q = P(\text{Failure}) = \frac{3}{4}$$

Consider what happens when a Bernoulli trial is repeated several times. Now we can discuss the probability of a particular number of successes.

$$p = P(\text{Success}), \quad q = P(\text{Failure})$$

Suppose we repeat a Bernoulli trial 6 times and each time the trial is independent of the others. From the multiplication principle, what is the probability of getting SFSSSF?

$$P(\text{SFSSSF}) = p q p p p q = p^4 q^2$$

What is the probability of getting SSSFFS?

$$P(\text{SSSFFS}) = p p p q q p = p^4 q^2$$

What is the probability of having 4 successes in the 6 trials?

How many ways are there to choose 4 of the 6 trials to be successful? $C_{6,4}$

$$P(4 \text{ successes in } 6 \text{ trials}) = C_{6,4} p^4 q^2$$

Example 3: If a single die is rolled 6 times, what is the probability of getting four 5's?

Success: 1 roll a 5
 on 1 trial: $P(\text{success}) = p = \frac{1}{6}$
 $P(\text{Failure}) = q = \frac{5}{6}$

$n = \# \text{ of trials} = 6$
 $x = \# \text{ of successes} = 4$

$$P(X=4) = C_{6,4} p^4 q^2$$

$$= C_{6,4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^2$$

$$= 15 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^2$$

$$\approx 0.008039$$

Binomial experiments:

Now let's write all this down in a more precise way:

A sequence of experiments is called a sequence of Bernoulli trials, or a *binomial experiment*, if

1. The same experiment is repeated a fixed number of times. (Each repetition is called a trial.)
2. Only two outcomes are possible on each trial.
3. The probability of success p for each trial is a constant.
4. All trials are independent.

Probabilities in Bernoulli Trials:

If the probability of success is p and the probability of failure is q , then the probability of exactly x successes in n Bernoulli trials is

$$P(X=x) = C_{n,x} p^x q^{n-x}$$

$$P(X=x) = C_{n,x} p^x q^{n-x}, \text{ or equivalently, } P(X=x) = \binom{n}{x} p^x q^{n-x}.$$

The binomial formula:

For any natural number n ,

$$(a+b)^n = C_{n,0} a^n b^0 + C_{n,1} a^{n-1} b^1 + C_{n,2} a^{n-2} b^2 + \dots + C_{n,n} a^0 b^n$$

$$= \sum_{i=0}^n C_{n,i} a^{n-i} b^i$$

no
need to
know
this!

Notice the similarity between this formula and the formula for probabilities in a sequence of Bernoulli trials. This is why such a sequence is called a *binomial experiment*.

→ n = 10

Example 4: If a single fair die is rolled 10 times, what is the probability of

- Exactly three 4's?
- At most three 4's?
- At least one 4?

on 1 trial, success = a 4 is rolled
 $S = \{1, 2, 3, 4, 5, 6\}$.

So $p = P(\text{Success}) = \frac{1}{6}$
 $q = P(\text{Failure}) = \frac{5}{6}$

Ⓐ exactly 3 successes in 10 trials

$$P(X=3) = C_{10,3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

$$= 120 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

$$\approx \boxed{0.155}$$

Ⓑ at most 3 successes in 10 trials

See the page after next ;)

Often we need to select a random sample from a larger population. To do this, we would select each item/person in a separate trial, without replacement. (We do not want to select the same person twice). Because the selected individuals are not replaced, the population size decreases by 1 with each trial. For that reason, this process is not a binomial experiment. (The probability of success is not the same for all the trials.) Instead, the probabilities follow a *hypergeometric distribution*.

However, as long as the sample does not exceed 5% of the population, the distribution will be extremely close to the binomial distribution, and it is acceptable to use the binomial distribution as an approximation.

Example 5: In the United States, about 9% of people have blood type B+. If 20 people donate blood, what is the probability that exactly three of them are B+? At least three? At least 18?

1 trial = 1 person

Success: person has B⁺ blood.

Failure: person does not have B⁺

$$p = P(\text{success}) = 0.09$$

$$q = P(\text{failure}) = 1 - 0.09 = 0.91$$

Ⓐ exactly 3 are B⁺

$$P(X=3) = C_{20,3} (0.09)^3 (0.91)^{17}$$

$$= 1140 (0.09)^3 (0.91)^{17} \approx \boxed{0.1672}$$

Ⓑ see next page

Ex 5 cont'd:

(b) at least 18 have B⁺

$$\begin{aligned}P(X \geq 18) &= P(X=18) + P(X=19) + P(X=20) \\&= C_{20,18} (0.09)^{18} (0.91)^2 + C_{20,19} (0.09)^{19} (0.91)^1 \\&\quad + C_{20,20} (0.09)^{20} (0.91)^0 \\&= 190 (0.09)^{18} (0.91)^2 + 20 (0.09)^{19} (0.91) + 1 (0.09)^{20} (1) \\&= 2.362 \times 10^{-17} + 2.459 \times 10^{-19} + 1.216 \times 10^{-21} \\&\approx \boxed{2.386 \times 10^{-17}}\end{aligned}$$

(very, very unlikely to have 18⁺ with blood type B⁺ when only 9% of population has it)

(c) At least 3 successes

E: $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + \dots + P(X=20)$
Use complement!

Complement: E^c: $P(X < 3) = P(X=0) + P(X=1) + P(X=2)$

$$= C_{20,0} (0.09)^0 (0.91)^{20} + C_{20,1} (0.09)^1 (0.91)^{19} + C_{20,2} (0.09)^2 (0.91)^{18}$$

$$= 1 \cdot 1 \cdot (0.91)^{20} + 20(0.09)(0.91)^{19} + 190(0.09)^2(0.91)^{18}$$

$$= 0.15164 + 0.29996 + 0.28183 = 0.73343$$

Subtract from 1 to get P(E).

$$P(X \geq 3) = 1 - 0.73343 = \boxed{0.2666}$$

Example 4 continued: (Dice, success = rolling a 4)

ⓑ At most three successes $(n=10, p=\frac{1}{6}, q=\frac{5}{6})$
 $X = \text{the number of successes}$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= C_{10,0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + C_{10,1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9 + C_{10,2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8 + C_{10,3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

$$= 1 \cdot 1 \cdot \left(\frac{5}{6}\right)^{10} + 10 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^9 + 45 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^8 + 120 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

$$= 0.16151 + 0.32301 + 0.29071 + 0.15505$$

$$= 0.93028 \approx \boxed{0.9303}$$

ⓒ At least one success

$$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + \dots + P(X=10)$$

Kuki! Use complement instead:

$$\text{Prob of complement: } P(X=0) = C_{10,0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} = 1 \cdot 1 \cdot \left(\frac{5}{6}\right)^{10} = 0.16151$$

$$P(X \geq 1) = 1 - 0.16151 \approx \boxed{0.8385}$$

The binomial distribution:

We can now generalize Bernoulli trials and determine probability distributions.

In a binomial experiment with n trials and probability of success p , we can create a binomial distribution table and a histogram, with the variable x representing the number of successes.

$$E(X) = \mu = 0\left(\frac{8}{27}\right) + 1\left(\frac{12}{27}\right) + 2\left(\frac{6}{27}\right) + 3\left(\frac{1}{27}\right) = \frac{27}{27} = 1$$

Example 6: Suppose a fair die is rolled three times and a success is considered to be rolling a number divisible by 3. $n = 3$

$x = \#$ successes
in 3 trials

On one trial: $S = \{1, 2, \dots, 6\}$

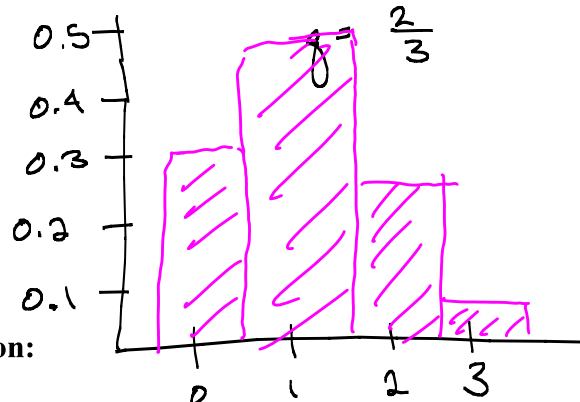
$E = \{3, 6\}$

$$p = P(E) = \frac{2}{6} = \frac{1}{3}$$

| x | $P(X=x)$ |
|-----|---|
| 0 | $C_{3,0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 = 1(1)\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ |
| 1 | $C_{3,1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9} = \frac{12}{27}$ |
| 2 | $C_{3,2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 = 3\left(\frac{1}{9}\right)\left(\frac{2}{3}\right) = \frac{2}{9} = \frac{6}{27}$ |
| 3 | $C_{3,3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 = 1\left(\frac{1}{3}\right)^3 (1) = \frac{1}{27}$ |

$$\text{Sum} = \frac{8}{27} + \frac{12}{27} + \frac{6}{27} + \frac{1}{27} = \frac{27}{27} = 1 \checkmark$$

Mean and standard deviation of the binomial distribution:



We can determine the mean number of successes (or the *expected value* of x) for any binomial distribution.

Mean (Expected Value) of the Binomial Distribution:

In a binomial experiment with n trials and probability of success p , the expected value, or mean, is

$$E(x) = \mu = np.$$

Standard Deviation of the Binomial Distribution:

$$\sigma = \sqrt{np(1-p)} = \sqrt{npq}.$$

Example 7: Roll a single fair die 10 times. Consider rolling a 4 to be a success. Find the expected number of successes.

On 1 trial, $P(\text{success}) = p = \frac{1}{6}$

$$E(X) = \mu = np = 10\left(\frac{1}{6}\right) = \frac{10}{6} = 1\frac{2}{3} \approx 1.67 \quad q = 1 - \frac{1}{6} = \frac{5}{6}$$

Find std. dev. of the number of successes:

$$\sigma = \sqrt{npq} = \sqrt{10\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} \approx 1.179 \quad n = 10$$

Example 8: Suppose that a certain cancer treatment has been found to be effective in 63% of patients. If 300 patients undergo the treatment, in how many cases would you expect the treatment to be effective? What is the standard deviation of the variable representing the number of cases in which the treatment is effective?

$$E(X) = \mu = np = 300(0.63) = \boxed{189 \text{ patients expect to be effective}}$$

$$\text{Std. dev. } \sigma = \sqrt{npq} = \sqrt{300(0.63)(0.37)} \approx \sqrt{69.93} \approx \boxed{8.36}$$

$$\begin{aligned} n &= 300 \\ p &= 0.63 \\ q &= 1 - 0.63 = 0.37 \\ \text{variance: } \sigma^2 &= \boxed{69.93} \\ &= npq \end{aligned}$$

Example 9: An exam has 10 true-false questions. Suppose that you haven't studied all semester, so your only choice is to guess randomly on each question. If you need a 70% to pass, what is your probability of passing?

Example 10: Suppose the test is multiple choice test with 5 choices on each of 10 questions. If you guess randomly at the answers, what is the probability of passing with at least a 70%? What grade would you expect to receive on the test?