4.1: Probability Basics
4.2: Events

An *experiment* is an activity with observable results. An experiment that does not always give the same result, even under the same conditions, is called a *probability experiment*, or a *random experiment*. Repetitions of an experiment are called *trials*.

> Examples: rolling dice, drawing cards, counting the number of defective radios in a sample from a shipment, using Excel to generate a random number, measuring the height of a plant, etc.

In probability theory, we'll usually use the word *experiment* to mean a *probability experiment.*

For a given experiment, we can make a list of outcomes of the experiment, called *simple events*, such that in each trial, one and only one of the simple events will occur. The set of all such simple events (outcomes) is called the *sample space*.

Any subset of the sample space is called an *event*. If such a subset contains more than one element of the sample space, we call the subset a *compound event*. If an event $E = \varnothing$, we call the event E an *impossible event*. If S is the sample space and $E = S$, then E is a *certain event*.

Example 1: We roll a single six-sided die. What is the sample space?

$S = \{1, 2, 3, 4, 5, 6\}$ $\quad n(S) = 6$

Example 2: We spin an American roulette wheel, which has pockets labeled 00, 0, 1, 2, 3, …, 35, 36. What is the sample space?

$S = \{00, 0, 1, 2, 3, \ldots, 33, 34, 35, 36\}$. $\quad n(S) = 38$

Example 3: The manager of a local cinema records the number of patrons attending a particular movie. The theater has a seating capacity of 500.

a. What is an appropriate sample space for this experiment?

$S = \{0, 1, 2, 3, \ldots, 500\}$

b. Write the event E that fewer than 50 people attend the movie.

$E = \{0, 1, 2, 3, \ldots, 49\}$

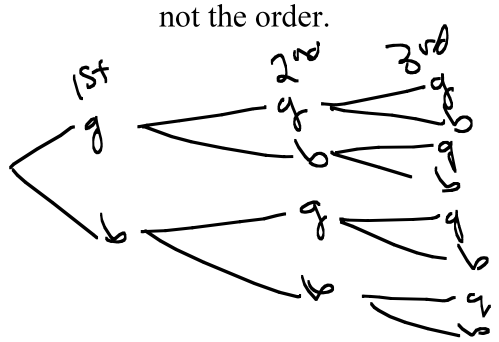
c. Write the event F that the theater is more than half full at the movie.

$F = \{251, 252, \ldots, 500\}$

Example 4: An experiment consists of studying the composition of a three-child family.

a. Write an appropriate sample space for this experiment if we are interested in the genders of the children in the order of their births.
b. Write the event *E* that there are two girls and a boy in the family.
c. Write the event *F* that the oldest child is a girl.
d. Write the event *G* that the oldest child is a girl and the youngest is a boy.
e. Write an appropriate sample space if we are only interested in the number of girls and not the order.

g: girl
b: boy



(a) $S = \{ggg, ggb, gbg, bgg, gbb, bgb, bbg, bbb\}$

(b) $E = \{ggb, gbg, bgg\}$

(c) $F = \{ggg, gbg, ggb, gbb\}$

(d) $G = \{ggb, gbb\}$

(e) $S_2 = \{3, 2, 1, 0\}$

Probability:

Probability is a way of quantifying the likelihood that an event occurs. There are three ways of assigning probabilities:

Frequentist (empirical) approach: Collect data about the frequency of each outcome, and use these data to assign a probability to each outcome.

What is probability a person in Texas votes Democrat?
What is the probability a patient has an adverse reaction to a drug?

Theoretical (classical) approach: Assign probabilities using logic and reasoning.

What is the probability of rolling a 5 on a single dice roll?

Subjective approach: Assign probabilities based on assumptions, educated guesses, and estimates.

What is the probability my cat has a successful surgery?

Probability models:

A *probability model*, or *probability assignment*, is a function that associates each outcome with a probability.

Suppose $S = \{e_1, e_2, \ldots e_n\}$ is a sample space. It contains *n* simple events, or outcomes. To each outcome, we can assign a number $P(e_i)$, called the *probability of the event* e_i.

> <u>Rules for probability assignments</u>:
>
> - For each simple event, the probability of that event is a number between 0 and 1, inclusive. In other words, $0 \le P(e_i) \le 1$.
> - The probabilities of all the simple events in the sample space add up to 1. In other words, $P(e_1) + P(e_2) + \ldots + P(e_n) = 1$.
>
> Any probability assignment that meets these conditions is called an *acceptable probability assignment.*

A probability assignment that reflects the actual or expected percentage of times a simple event occurs is called *reasonable*. In other words, it is reasonable if it makes sense based on the real world.

<u>Example 5:</u> Roll a single die. $S = \{1, 2, 3, 4, 5, 6\}$

A probability assignment that is both acceptable and reasonable:

$P(1) = 1/6$ $P(4) = 1/6$
$P(2) = 1/6$ $P(5) = 1/6$
$P(3) = 1/6$ $P(6) = 1/6$

A probability assignment that is not acceptable:

$P(1) = 1/2$ $P(4) = 1/4$
$P(2) = 1/4$ $P(5) = 0$
$P(3) = 1/4$ $P(6) = 0$

(Probabilities add up to more than 1)

A probability assignment that is acceptable but not reasonable:

$P(1) = 1/3$ $P(4) = 0$
$P(2)$ $1/3$ $P(5) = 0$
$P(3) = 1/3$ $P(6) = 0$

OK.
Probabilities add up to 1
All probs. are between 0 and 1

Empirical probability assignments: (our book calls this the frequentist approach)

Empirical (relative frequency) probability approximation:

$$P(E) \approx \frac{\text{frequency of occurence of } E}{\text{total number of trials}} = \frac{f(E)}{n}$$

(The larger n is, the better the approximation.)

Example 6: Suppose a coin is flipped 1000 times, resulting in 603 heads and 397 tails. Use these results to assign empirical probabilities for the experiment of flipping that coin.

H: heads
T: Tails

$P(H) = \frac{603}{1000} = 0.603$

$P(T) = \frac{397}{100} = 0.397$

Example 7: During the Spring 2016 semester, Lone Star College – North Harris enrolled 10,124 female students, 6,437 male students, and 10 students of unknown gender. Use these data to create a probability assignment. If a student is randomly selected, what is the probability that student is a female?
Data from
http://www.lonestar.edu/images/Student_Demographics_Official_Day_Spring_2015.pdf.

	Frequency	Relative Frequency	
Male	6437	6437/16571 ≈ 0.388	38.8%
Female	10124	10124/16571 ≈ 0.611	61.1%
Unknown	10	10/16571 ≈ 0.000603	0.06%
	$n = 16571$		

$\boxed{P(\text{Female}) = 0.611}$

Theoretical (classical) probability assignments:

When assigning probabilities theoretically (using logic and reasoning instead of data), we often depend on the he *equal-likelihood assumption*. To use this assumption, we need to be working with a sample space in which all the outcomes, or simple events, are equally likely to occur (i.e., their probabilities are equal). If several choices are possible for an experiment's sample space, it is often best to choose one in which all outcomes are equally likely.

Equally likely assumption (or equal-likelihood assumption):

If all events in a sample space are equally likely to occur, the probability of each is $\frac{1}{n}$, where n is the number of simple events in the sample space.

The equally likely assumption results in a basic principle of probability.

Basic Probability Principle:

Let S be a sample space of equally likely outcomes, and let the event E be a subset of S. Then the probability that event E occurs is

$$P(E) = \frac{n(E)}{n(S)}.$$

Example 8: Using the theoretical (classical) approach, create a probability assignment for the flip of a single coin.

$S = \{H, T\}$ $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

How do theoretical probabilities and empirical probabilities relate to each other?

Law of Large Numbers:

As the number of repetitions of an experiment increases, the empirical probability of an event approaches the theoretical probability of the event.

Whether the probability assignment is created empirically or theoretically, the following principles apply.

Probability of an event E:

Given a sample space S and an acceptable probability assignment, then for any event E ($E \subset S$), the following rules apply:

- If $E = \varnothing$, then $P(E) = 0$. (impossible event)
- If E is a simple event, then $P(E)$ is given by the original probability assignment.
- If E is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in E.
- If $E = S$, then $P(E) = P(S) = 1$. (certain event)

Steps for finding $P(E)$:

1. Set up an appropriate sample space.
2. Assign acceptable probabilities to each simple event.
3. $P(E)$ is the sum of the probabilities of all the simple events in E.

Note on notation:
$E \subset S$ means that E is a subset of S (i.e., every element of E is also an element of S).
The *empty set* is the set with no elements. The empty set is denoted $\varnothing$.

Example 9: Consider the following data from the University of Houston. Data from http://www.uh.edu/ir/reports/facts-at-a-glance/facts-at-a-glance.pdf.

a. What is the probability that a randomly selected undergraduate is in the College of Business?

$$\frac{4885}{33\,404} \approx \boxed{0.146}$$

b. What is the probability that a randomly selected graduate student is in a health-related program?

$$\frac{34+43+79}{6296} = \boxed{0.025}$$

(about 2.5% of grad students)

UNIVERSITY of HOUSTON

Fall 2015 Facts

Student Enrollment

College	Under-Graduate	Post-Baccalaureate	Graduate	Special Professional	Total
Architecture	605	3	81		689
Business	4,885	200	1,020		6,105
Education	1,694	61	676		2,431
Engineering	3,467	292	1,312		5,071
HRM	946	8	86		1,040
Law			130	717	847
CLASS	10,608	347	1,081		12,036
NSM	3,946	284	975		5,205
Nursing	42	54	34		130
Optometry			43	393	436
Pharmacy		5	79	468	552
Social Work			405		405
Technology	5,221	142	374		5,737
Exploratory Studies	1,990	30			2,020
Total	33,404	1,426	6,296	1,578	42,704

health-related

denominator for (a)

denominator for (b)

HRM—Conrad N. Hilton College of Hotel and Restaurant Management

CLASS — The College of Liberal Arts and Social Sciences

NSM— The College of Natural Sciences and Mathematics

Example 10: Flip two fair coins. What is the probability of getting

a. Two tails?
b. At least one head?
c. Exactly one head?
d. Two heads or two tails?
e. Three tails?

$n(S) = 4$

$S = \{HH, HT, TH, TT\}$

These are all equally likely

ⓐ $A = \{TT\}$

$P(A) = \frac{n(A)}{n(S)} = \boxed{\frac{1}{4}}$

ⓑ $B = \{HH, HT, TH\}$

$P(B) = \frac{n(B)}{n(S)} = \boxed{\frac{3}{4}}$

ⓒ $C = \{HT, TH\}$

$P(C) = \frac{n(C)}{n(S)} = \frac{2}{4} = \boxed{\frac{1}{2}}$

ⓓ $D = \{HH, TT\}$

$P(D) = \frac{2}{4} = \boxed{\frac{1}{2}}$

ⓔ $E = \emptyset$ (not possible)

$P(E) = \boxed{0}$

> Important Note: For any event E,
>
> $$0 \le P(E) \le 1.$$
>
> If $P(E) = 0$, then E is an *impossible event*.
> If $P(E) = 1$, then E is an *certain event*.

Example 11: Draw a single card from a standard card deck. What is the probability that it is

a. A spade?
b. A queen?
c. Black?
d. Higher than a 7? (Assume aces high.)

S = set of 52 cards

$n(S) = 52$

ⓐ $P(\text{Spade}) = \frac{n(\text{spades})}{n(S)} = \frac{13}{52} = \boxed{\frac{1}{4}}$

ⓑ $P(\text{Queen}) = \frac{n(\text{Queens})}{n(S)} = \frac{4}{52} = \boxed{\frac{1}{13}}$

ⓒ $P(\text{Black}) = \frac{n(\text{Black})}{n(S)} = \frac{26}{52} = \boxed{\frac{1}{2}}$

ⓓ $D = \{8, 9, 10, J, Q, K, A \text{ in each suit}\}$

$n(D) = 7\,(4 \text{ suits}) = 28$. So $P(D) = \frac{28}{52} = \boxed{\frac{7}{13}}$

Example 12: Roll a single die. What is the probability of rolling

a) a 5?
b) a prime number?
c) a multiple of 3?
d) a number larger than 10?
e) a number smaller than 10?

$S = \{1, 2, 3, 4, 5, 6\}$

a) $A = \{5\}$

$P(A) = \frac{n(A)}{n(S)} = \boxed{\frac{1}{6}}$

b) $B = \{2, 3, 5\}$

$P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \boxed{\frac{1}{2}}$

c) $C = \{3, 6\}$

$P(C) = \frac{n(C)}{n(S)} = \frac{2}{6} = \boxed{\frac{1}{3}}$

d) $P(D) = 0$
(impossible)

e) $E = \{1, 2, 3, 4, 5, 6\}$

$P(E) = \frac{n(E)}{n(S)} = \frac{6}{6} = \boxed{1}$

Example 13: Jennifer's electricity went out last night. Not having a flashlight, she had to choose a T-shirt at random in the dark. She owns 10 white T-shirts, 5 gray ones, 6 black ones and 2 red ones. What is the probability that she chose a gray T-shirt?

$n(S) = 10 + 5 + 6 + 2 = 23$

$P(\text{Gray}) = \boxed{\frac{5}{23}}$

Example 14: Suppose that Joe, Steve, Suzy, and Lisa work for the same company. The company wants to send two representatives to a particular conference and needs two to stay home and take care of the customers. All of them want to attend the conference, so they decide to put their names in a hat and draw two at random. What is the probability that Suzy and Lisa are selected?

$S = \{JoSt, JoSu, JoLi, StSu, StLi, SuLi\}$

$n(S) = 6$

$E = \{SuLi\}$

$P(E) = \frac{n(E)}{n(S)} = \boxed{\frac{1}{6}}$

b) What is the prob. that Joe is chosen?

$B = \{JoSt, JoSu, JoL\}$

$P(B) = \frac{3}{6} = \boxed{\frac{1}{2}}$

Example 15: The age distribution of Lone Star College – North Harris students during Spring 2015 is given in the table below. Data from http://www.lonestar.edu/images/Student_Demographics_Official_Day_Spring_2015.pdf.

Age Range	LSC-North Harris*	
	#	%
Under 20	4,183	25%
20-24	5,525	33%
25-29	2,425	15%
30-39	2,677	16%
40-49	1,179	7%
50+	581	4%
Unknown	1	0%
Grand Total	**16,571**	**100%**

What is the probability that a randomly selected student is

a) under 20 years old? $\frac{4183}{16571} \approx \boxed{0.25}$

b) Over 39? $0.07 + 0.04 = \boxed{0.11}$ ← using given percentages

or $\frac{1179 + 581}{16571} = \frac{1760}{16571} \approx \boxed{0.106}$ ← (more accurate)

c) Over 60?

We don't know, because the people aged 50-60 and those age 60^+ are combined into 1 category. We do know $P(\text{age} > 60) \le \frac{581}{16571}$, so $\boxed{P(\text{age } 60) \le 0.035}$

Example 16: Suppose a family has three children. What is the probability all the children are boys? What is the probability at least one child is a boy? Assume boys and girls are equally likely.

b = boy
g = girl

$S = \{bbb, bbg, bgb, gbb, ggb, gbg, bgg, ggg\}$

$n(S) = 8$. All 8 outcomes are equally likely.

E: all children are boys.

$E = \{bbb\}$

$P(E) = \frac{n(E)}{n(S)} = \boxed{\frac{1}{8}}$

F: At least 1 child is a boy.

$F = \{bbb, bbg, bgb, gbb, ggb, gbg, bgg\}$

$P(F) = \frac{n(F)}{n(S)} = \boxed{\frac{7}{8}}$

Skip odds

Odds:

Sometimes the likelihood (or unlikelihood) of an event is described using *odds* instead of probabilities.

Summary:

Probability: The event is contrasted against the whole.

Odds: The event is contrasted against the complement.

Converting from probability to odds:

From Probability to Odds:

- Odds for $E = \dfrac{P(E)}{P(E')}$
- Odds against $E = \dfrac{P(E')}{P(E)}$

When possible, express odds as ratios of whole numbers.

Example 17: What are the odds against rolling an ace when drawing a single card from a standard deck?

Example 18: Suppose that, based upon genetics, a child has a 0.08 probability of developing a certain disease. What are the odds against the child developing the disease?

Converting odds to probability:

From Odds to Probability:
If odds for an event E are $\frac{m}{n}$, (i.e. m:n) then $P(E)=\frac{m}{m+n}$.

Example 19: If the odds against a horse winning a race are 7:1, what is the probability that the horse will win?

Example 20: Suppose an insurance company has used past flood data to determine that determined that the odds against a particular house flooding are 150:1. What is the probability that the house floods?